

# Multi-soliton solutions in the chiral quark soliton model

Nobuyuki Sawado and Noriko Shiiki

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In this article a series of solutions with higher baryon numbers in the chiral quark soliton model are reported. *The chiral quark soliton model* (CQSM) is a simple quark model that incorporates the basic features of QCD, e.g. the chiral symmetry and its breakdown accompanied by the appearance of the Goldstone bosons. It was shown that the baryon number one ( $B = 1$ ) solution provides correct observable as a nucleon including mass, electromagnetic value, spin carried by quarks, parton distributions and octet, decuplet  $SU(3)$  baryon spectra. The  $B = 2$  axially symmetric soliton solution was obtained numerically by one of us (N. Sawado). For  $B \geq 3$ , in a series of our papers we obtained  $B = 3 \sim 9$ , 17 minimal energy soliton solutions with point-like symmetries and  $B = 3 \sim 5$  axially symmetric saddle-point solutions, using rational map ansatz constructed for multi-baryon number skyrmions. An interesting property of the solutions is that the symmetry of the background configuration is reflected in the degeneracy of the valence quark spectra. For instance, the resultant quark bound spectra are doubly degenerate for the axisymmetric solitons and are triply degenerate for  $B = 3$  tetrahedrally symmetric solitons. These results confirm the existence of the quark shell structure. The shells consist of the four-fold degenerate ground state and higher levels with various patterns, which are realized by the interplay of two symmetries,  $SU(2)_L \times SU(2)_R$  symmetry for the quarks and the symmetries of the chiral fields. To obtain correct physical observable, the quantum corrections are necessary. We shall show the quantum states of the axisymmetric solitons within the collective quantization. Upon quantization, various observable spectra of the chiral solitons are obtained. According to the Finkelstein-Rubinstein constraints, the quantum numbers of the solitons coincide with the physical observations only for  $B = 2$  and 4 while  $B = 3$  and 5 do not. The  $SU(3)$  extension of the  $B = 2$  soliton is also studied to predict various strange dibaryon states within this model.

## 1 Introduction

Although QCD is generally accepted as the underlying theory of the strong interaction, most low- and medium-energy nuclear phenomenology may be successfully

described in terms of the hadronic degrees of freedom. Investigations for deuteron photodisintegration and deep inelastic scattering of leptons by nuclei suggest the necessity of including quark degrees of freedom [1, 2, 3, 4, 5]. For this reason it was suggested that nuclear theory should be reformulated to take into account the underlying quark theory. However, QCD is too hard to get insight into the low- and medium-energy nuclear phenomenology since the coupling constant become large at these energy scales and one can not perform perturbation as in the other gauge theories. It is therefore necessary to formulate low-energy effective theories for the strong interaction.

The chiral quark soliton model is one of such QCD effective theories including quark degrees of freedom and baryons as a chiral soliton. In 1997 Diakonov *et al* predicted an exotic state with strangeness +1 within the topological soliton picture of baryons [6], and remarkably such resonance state has been experimentally discovered recently [7]. In a naive point of view, this new exotic baryon can be interpreted as a five quark bound state, opening up a new paradigm of the nuclei as a multi-quark bound state.

*The chiral quark soliton model* (CQSM) was developed in 1980's as a low-energy effective theory of QCD. Since it includes the Dirac sea quark contribution and explicit valence quark degrees of freedom, the model interpolates between the constituent quark model and the Skyrme model [8, 9, 10, 11, 12]. The CQSM is derived from the instanton liquid model of QCD vacuum and incorporates the nonperturbative feature of the low-energy QCD, spontaneous chiral symmetry breaking. It has been shown that the  $B = 1$  solution provides correct observable as a nucleon including mass, electromagnetic value, spin carried by quarks, parton distributions and octet  $SU(3)$  baryon spectra.

For  $B = 2$ , the stable axially symmetric soliton solution was found in Eq. [13]. The solution exhibits doubly degenerate bound spectra of the quark orbits in the background of the axially symmetric chiral field with winding number two. Upon quantization, various dibaryon spectra were obtained, showing that the quantum numbers of the ground state coincides with those of a physical deuteron [14, 15]. For  $B \geq 3$ , the Skyrme model shows that the soliton solutions have discrete, crystal-like symmetries [16, 17]. From the similarity of the chiral field action between the Skyrme model and the CQSM one can expect that soliton solutions in the CQSM have same symmetries as skyrmions with the same baryon number. Since it is too complicated to perform a numerical computation if one imposes such discrete symmetries directly on the chiral fields, Houghton, Manton and Sutcliffe thus proposed remarkable ansatz, rational map ansatz, for multi-skyrmions [18]. Applying this ansatz to the CQSM we obtained multi-chiral quark soliton solutions with point-like symmetries for  $B = 3 \sim 9, 17$  as well as saddle-point solutions for  $B = 5, 9$  [19, 20]. The solutions exhibit a large degeneracy and mass gap in the valence quark orbits. These results confirm the existence of the quark shell structure. Especially the large degeneracy implies that our solutions with such polyhedral symmetries may

be the lowest-lying configuration.

The solitons that we obtain are classical objects and therefore must be quantized to assign definite spin and isospin to them. Quantization of the solitons can be performed semiclassically for their rotational zero modes. Quantizing the solutions with discrete symmetries is, however, a formidable task in CQSM. Thus, before embarking those discrete symmetries, it will be instructive to study axially symmetric solutions which are much simpler [21]. Besides, considering the fact that for some higher baryon numbers, the ground states of the skyrmions do not agree with the experimental observation [22], the possibility that axially symmetric solutions may provide correct ground states can not be excluded. In fact it was found in Ref. [23] that the axially symmetric BPS monopoles up to charge five have lower energies than those of discrete symmetries. We therefore investigate classical and quantum multi-soliton solutions in the CQSM with axial symmetry up to  $B = 5$ .

Since the first prediction of the H-particle in a MIT bag model calculation [24], there have been many efforts to study the spectrum of dibaryonic systems including strangeness. We shall apply our formulation to study these six quark states constituting a dibaryon and make a prediction for its mass spectra.

In Sec. 2 the formulation of the chiral quark soliton model is introduced. In Sec. 3 we obtain axially symmetric soliton solutions for  $B = 2 \sim 5$  and solutions with the polyhedral symmetries for  $B = 3 \sim 9$  including saddle-point configurations. The relation between the symmetries of the chiral fields and the degeneracy of the valence quark spectra is discussed in Sec. 4 from a group theoretical point of view. In Sec. 5 we perform zero mode quantization for the obtained classical solitons. Imposing the Finkelstein-Rubinstein constraints on the states, the ground states of the axially symmetric solitons are constructed and examined if they agree with the experimental observation. Conclusions and discussions are in Sec.6.

## 2 The Chiral Quark Soliton Model : General Formalism

The CQSM is derived from the instanton liquid model of the QCD vacuum and incorporates the nonperturbative feature of the low-energy QCD, spontaneous chiral symmetry breaking. The vacuum functional is defined by Ref. [8]

$$\mathcal{Z} = \int \mathcal{D}\pi \mathcal{D}\psi \mathcal{D}\psi^\dagger \exp \left[ i \int d^4x \bar{\psi} (i\not{\partial} - MU^{\gamma_5}) \psi \right] \quad (1)$$

where the SU(2) matrix

$$U^{\gamma_5} = \frac{1 + \gamma_5}{2} U + \frac{1 - \gamma_5}{2} U^\dagger \quad \text{with} \quad U = \exp(i\boldsymbol{\tau} \cdot \boldsymbol{\pi}/f_\pi)$$

describes chiral fields,  $\psi$  is quark fields and  $M$  is the constituent quark mass.  $f_\pi$  is the pion decay constant and experimentally  $f_\pi \sim 93\text{MeV}$ .

The  $B = 1$  soliton solution has been studied in detail at classical and quantum level in Refs. [8, 9, 10, 11, 12]. To obtain solutions with  $B > 1$ , we shall employ the chiral fields with winding number  $B$  in the Skyrme model as the background of quarks, which can be justified as follows.

In Eq. (1), performing the functional integral over  $\psi$  and  $\psi^\dagger$  fields, one obtains the effective action

$$S_{\text{eff}}(U) = -iN_c \text{Sp} \ln iD = -iN_c \log \det iD, \quad (2)$$

where  $iD = i\cancel{\partial} - MU^{\gamma_5}$  is the Dirac operator. The classical solutions can be obtained by the extremum condition of (2) with respect to  $U$ . For this purpose, let us consider the derivative expansion of the action [25, 26, 12]. Up to quartic terms, we have,

$$S_{\text{eff}}(U) = \int d^4x \left[ -C \text{tr}(L_\mu L^\mu) + \frac{N_c}{32\pi^2} \text{tr} \left\{ \frac{1}{12} [L_\mu, L_\nu]^2 - \frac{1}{3} (\partial_\mu L^\mu)^2 + \frac{1}{6} (L_\mu L^\mu)^2 \right\} \right], \quad (3)$$

where  $L_\mu = U^\dagger \partial_\mu U$ . The baryon number  $B$  can be calculated in terms of the topological charge,

$$B = -\frac{1}{24\pi^2} \epsilon_{ijk} \int d^3x \text{tr}(L_i L_j L_k). \quad (4)$$

Suitably adjusting the coefficients  $\mathcal{C}$ , one can identify the first two terms of Eq. (3) with the Skyrme model action. However, the 4th order terms tend to destabilize solutions and no stable classical solution can be obtained from the above action [25, 27]. Nevertheless, because of their similarity, it will be justified to adopt the configurations of the solutions in the Skyrme model to chiral fields in the CQSM.

In the CQSM, the number of valence quark is associated with the baryon number such that the baryon number  $B$  soliton consist of  $N_c \times B$  valence quarks. If the quarks are strongly bound inside the soliton, their binding energy become large and the valence quarks can not be observed as positive energy particles [28, 29]. Thus, one gets the picture of the topological soliton model in the sense that the baryon number coincide with the winding number of the background chiral field when the valence quarks occupy all the levels diving into negative energy region.

Let us rewrite the effective action in Eq. (2) as

$$S_{\text{eff}} = -iN_c \log \det(i\cancel{\partial} - MU^{\gamma_5}) = -iN_c \log \det(i\partial_t - H(U^{\gamma_5})) \quad (5)$$

where

$$H(U^{\gamma_5}) = -i\alpha \cdot \nabla + \beta MU^{\gamma_5}. \quad (6)$$

The classical energy of the soliton can be estimated from the quark determinant in Eq. (5) [30, 31]. We introduce the eigenstates of operators,  $i\partial_t - H(U^{\gamma_5})$  and  $H(U^{\gamma_5})$ , such that

$$H(U^{\gamma_5})\phi_\mu(\mathbf{x}) = E_\mu\phi_\mu(\mathbf{x}), \quad (7)$$

$$(i\partial_t - H(U^{\gamma_5}))\Psi_{\mu,n} = \lambda_{\mu,n}\Psi_{\mu,n}, \quad (8)$$

where  $\Psi_{\mu,n} = e^{-i\omega_n t} \phi_\mu$  and  $\lambda_{\mu,n} = -E_\mu + \omega_n$ . Imposing on  $\Psi_{\mu,n}$  the anti-periodicity condition,  $\Psi_{\mu,n}(\mathbf{x}, T) = -\Psi_{\mu,n}(\mathbf{x}, 0)$ , reads

$$\omega_n T = (2n + 1)\pi. \quad (9)$$

The determinant in Eq. (5) then becomes

$$\begin{aligned} \det(i\partial_t - H) &= \prod_{\mu,n} \lambda_{\mu,n} = \prod_{\mu,n} \left( -E_\mu + \frac{(2n+1)\pi}{T} \right) \\ &= C \prod_{\mu,n \geq 0} \left( 1 - \frac{|E_\mu|^2 T^2}{(2n+1)^2 \pi^2} \right) = C \prod_{\mu} \cos\left(\frac{1}{2}|E_\mu|T\right) \\ &= \frac{C}{2} \exp\left(i\frac{1}{2} \sum_{\mu} |E_\mu|T\right) \prod_{\mu} \left(1 + \exp(-i|E_\mu|T)\right) \end{aligned} \quad (10)$$

where

$$C = \prod_{n \geq 0} \left( -\frac{(2n+1)^2 \pi^2}{T^2} \right)$$

and the product formula for the cosine function  $\cos(z) = \prod_{n \geq 1} (1 - 4z^2/(2n-1)^2 \pi^2)$  has been used. Inserting (10) into Eq. (5), one obtains

$$S_{\text{eff}} = -N_c T \sum_{\mu} n_{\mu} |E_{\mu}| + N_c T \frac{1}{2} \sum_{\mu} |E_{\mu}|, \quad (11)$$

where  $n_{\mu}$  is the valence quark occupation number which takes values only 0 or 1. Correspondingly, the classical energy is given by

$$E_{\text{static}} = E_{\text{val}} + E_{\text{vac}} \quad (12)$$

where

$$E_{\text{val}} = N_c \sum_{\mu} n_{\mu} |E_{\mu}|, \quad E_{\text{vac}} = -\frac{1}{2} N_c \sum_{\mu} |E_{\mu}|,$$

representing the valence quark and sea quark contribution to the total energy respectively.

The effective action  $S_{\text{eff}}(U)$  is ultraviolet divergent and hence must be regularized. Using the proper-time regularization scheme [32], we can write

$$\begin{aligned} S_{\text{eff}}^{\text{reg}}[U] &= \frac{i}{2} N_c \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \text{Sp} \left( e^{-D^\dagger D \tau} - e^{-D_0^\dagger D_0 \tau} \right) \\ &= \frac{i}{2} N_c T \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \text{Sp} \left[ e^{-\tau(H^2 + \omega^2)} - e^{-\tau(H_0^2 + \omega^2)} \right] \end{aligned} \quad (13)$$

where  $D_0$  and  $H_0$  are operators with  $U = 1$ . The total energy is then given by

$$E_{\text{static}}[U] = E_{\text{val}}[U] + E_{\text{vac}}[U] - E_{\text{vac}}[U = 1] \quad (14)$$

where

$$E_{\text{val}} = N_c \sum_i E_{\text{val}}^{(i)}, \quad E_{\text{vac}} = N_c \sum_\mu \left\{ \mathcal{N}(E_\mu) |E_\mu| + \frac{\Lambda}{\sqrt{4\pi}} \exp \left[ - \left( \frac{E_\mu}{\Lambda} \right)^2 \right] \right\}$$

with

$$\mathcal{N}(E_\mu) = -\frac{1}{\sqrt{4\pi}} \Gamma \left( \frac{1}{2}, \left( \frac{E_\mu}{\Lambda} \right)^2 \right)$$

and  $E_{\text{val}}^{(i)}$  is the valence energy of the  $i$  th valence quark.  $\Lambda$  is a cutoff parameter evaluated by the condition that the derivative expansion of Eq. (13) reproduces the pion kinetic term with the correct coefficient, *i.e.*,

$$f_\pi^2 = \frac{N_c M^2}{4\pi^2} \int_{1/\Lambda^2}^\infty \frac{d\tau}{\tau} e^{-\tau M^2}. \quad (15)$$

In this model, the constituent quark mass  $M$  is the only free parameter and we take the value  $M = 400$  MeV, in which the observable of the nucleon and the delta are well reproduced [11]. From Eq. (15) and by using the values of  $M, f_\pi$ , we obtain  $\Lambda \sim 637$  MeV.

For  $B = 1$ , one imposes a spherically symmetric ansatz (*hedgehog ansatz*)

$$U(\mathbf{r}) = \exp(iF(r)\hat{\mathbf{r}} \cdot \boldsymbol{\tau}) = \cos F(r) + i\hat{\mathbf{r}} \cdot \boldsymbol{\tau} \sin F(r), \quad (16)$$

with the boundary condition for the profile function  $F(r)$

$$F(0) = -\pi, \quad F(\infty) = 0. \quad (17)$$

Substituting the ansatz (16) into Eq. (4) under the boundary condition (17), one gets

$$B = \frac{1}{2\pi^2} \int_0^\infty \frac{\sin^2 F(r)}{r^2} \frac{dF(r)}{dr} 4\pi r^2 dr = \frac{1}{\pi} \left[ F + \frac{\sin 2F}{2} \right]_{-\pi}^0 = 1. \quad (18)$$

The one-quark hamiltonian (6) becomes

$$H(U^{\gamma_5}) = -i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta M(\cos F(r) + i\gamma_5 \hat{\mathbf{r}} \cdot \boldsymbol{\tau} \sin F(r)). \quad (19)$$

This hamiltonian does not commute with the total angular momentum  $\mathbf{J}$  nor the isospin  $\boldsymbol{\tau}/2$  but commute with the grand spin operator  $\mathbf{K} = \mathbf{J} + \boldsymbol{\tau}/2$ .  $H$  also commutes with the parity operator  $\mathcal{P} = \gamma_0$ . Hence the one-quark eigenstates are labeled by the  $K = 0, 1, 2, \dots$  and the parity  $\mathcal{P} = \pm$ . The three valence quarks

occupy their lowest states  $K^{\mathcal{P}} = 0^+$  and responsible for the baryon number of the soliton ( $= 1$ ). In this context, the baryon number is not a topological origin unlike the Skyrme @model.

Field equations for the chiral fields can be obtained by demanding that the total energy in Eq. (14) be stationary with respect to variation of the profile function  $F(r)$ ,

$$\frac{\delta}{\delta F(r)} E_{\text{static}} = 0 ,$$

which produces

$$S(r) \sin F(r) = P(r) \cos F(r), \quad (20)$$

where

$$S(r) = N_c \sum_{\mu} (n_{\mu} \theta(E_{\mu}) + \text{sign}(E_{\mu}) \mathcal{N}(E_{\mu})) \langle \mu | \gamma^0 \delta(|x| - r) | \mu \rangle , \quad (21)$$

$$P(r) = N_c \sum_{\mu} (n_{\mu} \theta(E_{\mu}) + \text{sign}(E_{\mu}) \mathcal{N}(E_{\mu})) \langle \mu | i \gamma^0 \gamma^5 \hat{\mathbf{r}} \cdot \boldsymbol{\tau} \delta(|x| - r) | \mu \rangle . \quad (22)$$

The procedure to obtain self-consistent solutions of Eq. (20) is that 1) solve the eigenequation in the hamiltonian (19) under an assumed initial profile function  $F_0(r)$ , 2) use the resultant eigenfunctions and eigenvalues to calculate  $S(r)$  and  $P(r)$ , 3) solve (20) to obtain a new profile function, 4) repeat 1) – 3) until the self-consistency is attained.

The calculated energy of the  $B = 1$  soliton is  $E_{\text{static}} = 1192$  MeV with the constituent quark mass  $M = 400$  MeV.

### 3 The Classical Configurations

#### 3.1 The Axially Symmetric Configuration

It is known that the minimal energy solution for  $B > 1$  is not spherically symmetric. In the Skyrme model the configuration with  $B = 2$  is axially symmetric [33, 34, 35, 36] and can be written by

$$U(\mathbf{x}) = \cos F(\rho, z) + i \boldsymbol{\tau} \cdot \hat{\mathbf{n}} \sin F(\rho, z), \quad (23)$$

where

$$\hat{\mathbf{n}} = (\sin \Theta(\rho, z) \cos m_w \varphi, \sin \Theta(\rho, z) \sin m_w \varphi, \cos \Theta(\rho, z)) \quad (24)$$

and  $m_w$  is the winding number of the pion fields. We shall use this configuration in the background to obtain axially symmetric chiral quark solitons.

The extremum conditions for the total energy

$$\frac{\delta}{\delta F(\rho, z)} E_{\text{static}}[U] = 0, \quad \frac{\delta}{\delta \Theta(\rho, z)} E_{\text{static}}[U] = 0 \quad (25)$$

yield the following equations of motion for the profile functions,

$$R^T(\rho, z) \cos \Theta(\rho, z) = R^L(\rho, z) \sin \Theta(\rho, z), \quad (26)$$

$$S(\rho, z) \sin F(\rho, z) = P(\rho, z) \cos F(\rho, z) \quad (27)$$

where

$$P(\rho, z) = R^T(\rho, z) \sin \Theta(\rho, z) + R^L(\rho, z) \cos \Theta(\rho, z). \quad (28)$$

In terms of eigenfunction  $\phi$  in Eq. (7),  $R^T, R^L$  and  $S$  are given by

$$R^T(\rho, z) = R_{\text{val}}^T(\rho, z) + R_{\text{vac}}^T(\rho, z), \quad (29)$$

$$R^L(\rho, z) = R_{\text{val}}^L(\rho, z) + R_{\text{vac}}^L(\rho, z), \quad (30)$$

$$S(\rho, z) = S_{\text{val}}(\rho, z) + S_{\text{vac}}(\rho, z) \quad (31)$$

where

$$R_{\text{val}}^T(\rho, z) = \sum_i \int d\varphi \bar{\phi}_i(\rho, \varphi, z) i\gamma_5 (\tau_1 \cos m_w \varphi + \tau_2 \sin m_w \varphi) \phi_i(\rho, \varphi, z),$$

$$R_{\text{vac}}^T(\rho, z) = \sum_{\mu} \mathcal{N}(E_{\mu}) \text{sgn}(E_{\mu}) \int d\varphi \bar{\phi}_{\mu}(\rho, \varphi, z) i\gamma_5 \\ \times (\tau_1 \cos m_w \varphi + \tau_2 \sin m_w \varphi) \phi_{\mu}(\rho, \varphi, z),$$

$$R_{\text{val}}^L(\rho, z) = \sum_i \int d\varphi \bar{\phi}_i(\rho, \varphi, z) i\gamma_5 \tau_3 \phi_i(\rho, \varphi, z),$$

$$R_{\text{vac}}^L(\rho, z) = \sum_{\mu} \mathcal{N}(E_{\mu}) \text{sgn}(E_{\mu}) \int d\varphi \bar{\phi}_{\mu}(\rho, \varphi, z) i\gamma_5 \tau_3 \phi_{\mu}(\rho, \varphi, z),$$

$$S_{\text{val}}(\rho, z) = \sum_i \int d\varphi \bar{\phi}_i(\rho, \varphi, z) \phi_i(\rho, \varphi, z),$$

$$S_{\text{vac}}(\rho, z) = \sum_{\mu} \mathcal{N}(E_{\mu}) \text{sgn}(E_{\mu}) \int d\varphi \bar{\phi}_{\mu}(\rho, \varphi, z) \phi_{\mu}(\rho, \varphi, z),$$

and subscripts, vac and val, represent the vacuum and valence quark contributions respectively. The boundary conditions for the profile functions were constructed by Braaten and Carson [37];

$$F(\rho, z) \rightarrow 0 \quad \text{as} \quad \rho^2 + z^2 \rightarrow \infty, \\ F(0, 0) = -\pi, \quad \Theta(0, z) = \begin{cases} 0, & z > 0 \\ \pi, & z < 0 \end{cases}. \quad (32)$$



In Fig. 1, we show the spectrum of the quark orbits in the background of chiral fields with winding number  $m_w = 2, 4$ , as a function of the size parameter  $X$ . The axially symmetric profile functions are parameterized by  $X$  as

$$\begin{aligned} F(\rho, z) &= -\pi + \pi\sqrt{\rho^2 + z^2}/X \quad \text{for } \sqrt{\rho^2 + z^2} \leq X \\ &= 0 \quad \text{otherwise,} \end{aligned} \quad (33)$$

$$\Theta(\rho, z) = \tan^{-1}(\rho/z). \quad (34)$$

To examine the spectrum in detail, let us consider the hamiltonian defined in Eq. (6). For the axially symmetric chiral field in Eq. (23), this hamiltonian commutes with the third component of the grand spin operator  $K_3$  and the time-reversal operator  $T$ . These are specifically,

$$K_3 = L_3 + \frac{1}{2}\sigma_3 + \frac{1}{2}m_w\tau_3, \quad (35)$$

$$T = i\gamma_1\gamma_3 \cdot i\tau_1\tau_3C \quad (36)$$

where  $L_3$ ,  $\sigma_3$ , and  $\tau_3$  are respectively the third component of orbital angular momentum, spin, and isospin operator, and  $C$  is a charge conjugation operator. The parity operator is defined by  $P = \gamma_0$  for odd  $B$ , and  $P = \gamma_0\tau_3$  for even  $B$ . Thus, the eigenvalues of the hamiltonian can be specified by the magnitude of  $K_3$  and the parity  $\pi = \pm$ . We have  $K_3 = 0, \pm 1, \pm 2, \pm 3, \dots$  for odd  $B$ , and  $K_3 = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{7}{2}, \dots$  for even  $B$ . Since the hamiltonian is invariant under time reverse, the states of  $+K_3$  and  $-K_3$  are degenerate in energy.

From Fig. 1 it can be seen that for  $m_w = 2$ , the bound states diving into negative region are doubly degenerate with  $K_3 = \pm \frac{1}{2}$ . Thus putting  $N_c = 3$  valence quarks on each of the bound levels, we have the  $B = 2$  soliton solution. Similary, for  $m_w = 3$ , we obtained the spectrum of  $K_3 = \pm 1^-$  (double degeneracy) and  $K_3 = 0^+$  states diving into negative-energy region which corresponds to the  $B = 3$  soliton solution. For  $m_w = 4$ , the spectrum of  $K_3 = \pm \frac{1}{2}^+$  and  $K_3 = \pm \frac{3}{2}^-$  (both doubly degenerate) states dive into negative region, having the  $B = 4$  soliton solution. For  $m_w = 5$ , the spectrum of  $K_3 = \pm 2^+$  (double),  $K_3 = \pm 1^-$  (double) and  $K_3 = 0^+$  states dive into negative-energy region, having the  $B = 5$  soliton solution. These results confirm that the baryon number of the soliton is identified with the number of diving levels occupied by  $N_c$  valence quarks. It is interesting that the degeneracy which occurs due to symmetry of the chiral field reduces the number of states, making large shell gaps. This observation indicates that degeneracy in the valence quark spectra contributes to make classical energies of the soliton solutions lower. In fact, our  $B = 2$  solution which is considered to be the minimum energy soliton from the study of the  $B = 2$  skyrmion provides the maximum degeneracy in spectra. It will be interesting to study minimum solutions from this point of view.

Table 1: The classical observable, Mass (in MeV), the mean radius of toroid and the root mean square radius (in fm) for  $B = 1 \sim 5$ .

$B$	$E_{\text{val}}$					$E_{\text{vac}}$	$E_{\text{static}}$	$\langle \rho \rangle$	$\sqrt{\langle r^2 \rangle}$
1	173					674	1192		0.785
2	173	173				1166	2204	0.672	0.821
3	173	173	298			1561	3493	0.659	0.854
4	106	106	232	232		2727	4753	0.971	1.140
5	145	145	319	319	409	2537	6543	1.048	1.225

The baryon number density is defined by the zeroth component of the baryon current [9],

$$b(\mathbf{x}) = \frac{1}{N_c} \langle \bar{\psi} \gamma_0 \psi \rangle = b_{\text{val}}(\mathbf{x}) + b_{\text{vac}}(\mathbf{x}) \quad (37)$$

where

$$\begin{aligned} b_{\text{val}}(\mathbf{x}) &= \frac{1}{N_c} \sum_i \int d\varphi \phi_i^\dagger(\rho, \varphi, z) \phi_i(\rho, \varphi, z) \\ b_{\text{vac}}(\mathbf{x}) &= \frac{1}{N_c} \left[ \sum_\mu \mathcal{N}(E_\mu) \text{sgn}(E_\mu) \int d\varphi \phi_\mu^\dagger(\rho, \varphi, z) \phi_\mu(\rho, \varphi, z) \right. \\ &\quad \left. - \sum_\mu \mathcal{N}(E_\mu^{(0)}) \text{sgn}(E_\mu^{(0)}) \int d\varphi \phi_\mu^{(0)\dagger}(\rho, \varphi, z) \phi_\mu^{(0)}(\rho, \varphi, z) \right]. \end{aligned}$$

A contour plot of the baryon number density for each baryon number is shown in Fig. 4. It can be seen that they have toroidal in shape.

The mean radius  $\langle \rho \rangle$  is given by

$$\langle \rho \rangle = \langle \rho \rangle_{\text{val}} + \langle \rho \rangle_{\text{vac}} \quad (38)$$

where

$$\begin{aligned} \langle \rho \rangle_{\text{val}} &= \frac{1}{m_w} \sum_i \int \rho d\rho dz d\varphi \rho \phi_i^\dagger(\rho, \varphi, z) \phi_i(\rho, \varphi, z), \\ \langle \rho \rangle_{\text{vac}} &= \frac{1}{m_w} \sum_\mu \mathcal{N}(E_\mu) \text{sgn}(E_\mu) \int \rho d\rho dz d\varphi \rho \phi_\mu^\dagger(\rho, \varphi, z) \phi_\mu(\rho, \varphi, z). \end{aligned}$$

The root mean square radius is given by

$$\sqrt{\langle r^2 \rangle} = \sqrt{\langle r^2 \rangle_{\text{val}} + \langle r^2 \rangle_{\text{vac}}} \quad (39)$$

where

$$\begin{aligned}\langle r^2 \rangle_{\text{val}} &= \frac{1}{m_w} \sum_i \int \rho d\rho dz d\varphi (\rho^2 + z^2) \phi_i^\dagger(\rho, \varphi, z) \phi_i(\rho, \varphi, z), \\ \langle r^2 \rangle_{\text{vac}} &= \frac{1}{m_w} \sum_\mu \mathcal{N}(E_\mu) \text{sgn}(E_\mu) \int \rho d\rho dz d\varphi (\rho^2 + z^2) \phi_\mu^\dagger(\rho, \varphi, z) \phi_\mu(\rho, \varphi, z).\end{aligned}$$

These values for each baryon number are shown in Table 1.

### 3.2 Multi-Winding Number Configurations with Polyhedral Symmetries

In the Skyrme model it is known that minimal energy configurations with  $B \geq 3$  have discrete crystal-like symmetries [16] rather than axisymmetry. We expect that configurations of the CQSM inherits the same discrete symmetry as skyrmions. However, it is too complicated to perform a numerical computation if one imposes such discrete symmetries directly on the chiral fields. Thus Houghton, Manton and Sutcliffe proposed remarkable ansatz for the chiral fields, rational map ansatz [18]. According to this ansatz, the chiral fields are expressed in a rational map as

$$U(r, z) = \exp(iF(r)\hat{\mathbf{n}}_R \cdot \boldsymbol{\tau}), \quad (40)$$

where

$$\hat{\mathbf{n}}_R = \frac{1}{1 + |R(z)|^2} (2\text{Re}[R(z)], 2\text{Im}[R(z)], 1 - |R(z)|^2)$$

and  $R(z)$  is the rational map. The complex coordinate  $z$  is given by  $z = \tan(\theta/2)e^{i\varphi}$  via stereographic projection.

Rational maps are maps from  $CP(1)$  to  $CP(1)$  (equivalently, from  $S^2$  to  $S^2$ ) classified by winding number. In Ref. [18] Houghton *et al.* showed that  $B = N$  skyrmions can be well-approximated by rational maps with winding number  $N$ . The rational map with winding number  $N$  possesses  $(2N + 1)$  complex parameters whose values can be determined so as to realize minimal energies within assumed symmetries of the skyrmion. We shall use this ansatz for the background chiral

fields in the CQSM. Their explicit forms of the map are [18, 38]

$$\begin{aligned}
R_3 &= \frac{\sqrt{3}iz^2 - 1}{z(z^2 - \sqrt{3}i)}, \\
R_4 &= \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}, \\
R_5 &= \frac{z(z^4 + 3.94z^2 + 3.07)}{3.07z^4 - 3.94z^2 + 1}, \\
R_6 &= \frac{z^4 + 0.16i}{z^2(0.16z^4i + 1)}, \\
R_7 &= \frac{7/\sqrt{5}z^6 - 7z^4 - 7/\sqrt{5}z^2 - 1}{z(z^6 + 7/\sqrt{5}z^4 + 7z^2 - 7/\sqrt{5})}, \\
R_8 &= \frac{z^6 - 0.14}{z^2(0.14z^6 + 1)}, \\
R_9 &= \frac{z(-3.38 - 11.19iz^4 + z^8)}{1 - 11.19iz^4 - 3.38z^8}, \\
R_5^* &= \frac{z(z^4 - 5)}{-5z^4 + 1}, \\
R_9^* &= \frac{5i\sqrt{3}z^6 - 9z^4 + 3i\sqrt{3}z^2 + 1 - 1.98z^2(z^6 - i\sqrt{3}z^4 - z^2 + i\sqrt{3})}{z^3(-z^6 - 3i\sqrt{3}z^4 + 9z^2 - 5i\sqrt{3} - 1.98z(-i\sqrt{3}z^6 + z^4 + i\sqrt{3}z^2 - 1))}, \\
R_{17} &= \frac{17z^{15} - 187z^{10} + 119z^5 - 1}{z^2(z^{15} + 119z^{10} + 187z^5 + 17)}. \tag{41}
\end{aligned}$$

Field equations for the chiral fields can be obtained by demanding that the total energy in Eq. (14) be stationary with respect to variation of the profile function  $F(r)$ ,

$$\frac{\delta}{\delta F(r)} E_{\text{static}} = 0,$$

which produces

$$S(r) \sin F(r) = P(r) \cos F(r), \tag{42}$$

where

$$S(r) = N_c \sum_{\mu} (n_{\mu} \theta(E_{\mu}) + \text{sign}(E_{\mu}) \mathcal{N}(E_{\mu})) \langle \mu | \gamma^0 \delta(|x| - r) | \mu \rangle, \tag{43}$$

$$P(r) = N_c \sum_{\mu} (n_{\mu} \theta(E_{\mu}) + \text{sign}(E_{\mu}) \mathcal{N}(E_{\mu})) \langle \mu | i\gamma^0 \gamma^5 \hat{\mathbf{n}}_R \cdot \boldsymbol{\tau} \delta(|x| - r) | \mu \rangle. \tag{44}$$

Table 2: Mass spectra for  $B = 1 - 9, 17$  also for some excited states  $B = 5^*, 9^*$  (in MeV). The data for  $B = 2$  are taken from Ref. [14]. The ratio of the mass  $E_{\text{static}}$  to  $B \times E_{\text{static}}^{(B=1)}$  are compared to that of the Skyrme model [18].

$B$		$E_{\text{val}}^{(i)}$								$E_{\text{field}}$	$E_{\text{static}}$	$E_{\text{static}}/BE_{\text{static}}^{(B=1)}$	
												Ours	Skyrme
1	173									674	1192	1.00	1.00
2	173	173								1166	2204	0.92	0.95
3	210	210	210							1633	3522	0.98	0.96
4	144	146	146	146						2628	4378	0.92	0.92
5	123	131	131	139	210					3265	5467	0.92	0.93
6	120	124	150	150	206	206				3740	6603	0.92	0.92
7	115	120	120	120	166	166	166			4554	7478	0.90	0.90
8	97	97	115	120	139	139	203	203		5229	8565	0.90	0.91
9	69	101	104	104	107	166	166	179	179	6046	9573	0.89	0.90 <sub>6</sub>
17	83	95	95	95	153	156	157	173	175	10586	18650	0.93	0.88
	177	178	179	192	194	194	196	196					
5*	157	157	157	232	232					2874	5680	0.95	1.00
9*	99	105	105	121	142	142	210	210	210	5700	9742	0.91	0.91

The baryon density  $b(\mathbf{x})$  is defined by

$$b(\mathbf{x}) = \frac{1}{N_c} \langle \bar{\psi} \gamma_0 \psi \rangle = b_{\text{val}}(\mathbf{x}) + b_{\text{field}}(\mathbf{x}), \quad (45)$$

where

$$\begin{aligned}
b_{\text{val}}(\mathbf{x}) &= \sum_i b_{\text{val}}^{(i)}(\mathbf{x}) = \frac{1}{N_c} \sum_i \phi_i(\mathbf{x})^\dagger \phi_i(\mathbf{x}), \\
b_{\text{field}}(\mathbf{x}) &= \frac{1}{N_c} \left[ \sum_\mu \text{sign}(E_\mu) \mathcal{N}(E_\mu) \phi_\mu(\mathbf{x})^\dagger \phi_\mu(\mathbf{x}) \right. \\
&\quad \left. - \sum_\mu \text{sign}(E_\mu^{(0)}) \mathcal{N}(E_\mu^{(0)}) \phi_\mu^{(0)}(\mathbf{x})^\dagger \phi_\mu^{(0)}(\mathbf{x}) \right]. \quad (46)
\end{aligned}$$

To examine the shell structure of the quarks, we evaluate the radial density for the  $i$ th valence quark  $\rho^{(i)}(r)$  in which the angular degrees of freedom are integrated out, via,

$$\rho^{(i)}(r) = \int d\varphi \int \sin \theta d\theta \, b_{\text{val}}^{(i)}(r, \theta, \varphi) \quad (47)$$

with the baryon number

$$B = \sum_i \int dr r^2 \rho^{(i)}(r). \quad (48)$$

The profile functions for  $B = 3 - 9, 17$  are plotted in Fig. 5. In Table 2 are the results for the valence quark levels as well as the vacuum sea contributions. The valence quark spectra show various degenerate patterns depending on the background configuration. The results of the ratio of the mass  $E_{\text{static}}$  to  $B \times E_{\text{static}}^{B=1}$  and comparison to those of the Skyrme model are also in Table. 2. They are qualitatively in agreement. Using Eq. (46), we estimated the baryon number density (see Fig. 6). As expected the density inherits the same symmetry as the corresponding skyrmion.

The valence quark spectra for various  $B$  are shown in Fig. 7. These results strongly suggest the existence of shell structure for the valence quarks. The spectra show (i) four fold degeneracy of the ground state labeled by  $\mathcal{G}$  and various degenerate pattern for excited levels labeled by  $\mathcal{A}_1, \mathcal{A}_2, \dots$ , (ii) a large energy gap between the ground state  $\mathcal{G}$  and the first excited level  $\mathcal{A}_1$ . Small dispersions of the spectra are observed in the results. In some cases they are caused by the finite size effect of the basis (ex.  $B = 4$ ). Growing the size  $r_{\text{max}}$  and increasing the number of the basis, more accurate degeneracy will be attained.

In Fig. 8 are the results of  $\rho^{(i)}(r)$  for  $B = 3 - 9, 17$ . The behaviour of the density near the origin confirms the existence of three shells ( $\mathcal{G}, \mathcal{A}_1, \mathcal{A}_2$ ).  $\mathcal{G}$  behaves like “ $S$ -wave” and others like “ $P$ -,  $D$ -wave” in a hydrogen-like atom. However most of the densities are nearly on the same surface and very small (not zero) near the origin. The plateau in the density observed at the center of the nucleus [39] can not be attained in our solutions. Therefore one may need to employ the multi-shell ansatz [40] even in the case of light nuclei.

The solutions that we obtained here are totally classical one. To reach the physical observations, one need to take into account the quantum corrections. Especially, the absolute mass tends to be much higher than the real nucleon mass due to a lack of the Casimir effects– the loop corrections of the order of  $O(N_c^0)$ . For  $B = 1$ , some attempts for this subject have been done in the Skyrme model [41, 42, 43] and in the hybrid quark soliton model [44]. In both cases the calculated values properly reduce the large classical energy to the physical nucleon mass. For  $B > 1$ , because of the lower symmetry of the chiral fields, the estimations of these effects are tedious task. Only one work for the  $B = 2$  chiral soliton is reported in Ref. [45] in which the configuration is restricted to  $SO(3)$ . Since the quantum corrections strongly depend on  $B$  it is difficult to extract any information of the binding energy from our classical solutions.

### 3.3 Classical Dynamics of Slowly Moving $B = 2$ Solitons

So far our concern has been restricted to static solutions, we shall now extend our scope to cover dynamics. Once a multi-soliton solution is obtained, one can study the low-energy dynamics of the multi-solitons following the geodesic approximation [46]. The existence of multi-soliton solutions means that there is no net force between the solitons. Manton showed that the multi-BPS monopole solution is a consequence of the cancellation of the repulsive force by the exchange of gauge bosons and the

attractive force by exchange of Higgs bosons [47]. The cancellation between the forces can be attained when the solution saturates the Bogomol'nyi bound [48]. For the Skyrme model, one finds

$$E \geq \frac{12\pi^2 f_\pi}{e} B \quad (49)$$

where  $e$  is a dimensionless constant which can be fixed experimentally. It is known that skyrmions do not saturate this bound and have slightly higher energies.

If the solitons which are far apart initially are given some impact parameter and start to move slowly towards each other, they will trace a path close to the valley of the potential energy without climbing the wall of the potential barrier. The valley is just the parameter space of the soliton solution which is called a moduli space. If the solitons move slowly, or adiabatically, it will be a good approximation to consider their dynamics only on the moduli space (truncation). Then the time evolution of the solitons is approximately the geodesics on the moduli space.

In the  $B = 2$  rational map ansatz, by imposing axisymmetry we get

$$R_2^* = \frac{z^2 - a}{-az^2 + 1} \quad (50)$$

where  $a$  is a parameter of a geodesic in the moduli space with  $-1 \leq a \leq 1$ . For  $a = 0$ , one recovers the minimal energy toroidal configuration. Letting the parameter  $a$  be time-dependent, we can examine the adiabatic time evolution of initially far-apart two solitons, that is,

$$U(r, z; a) = U(r, z; a(t)). \quad (51)$$

The numerical solutions for various values of  $a$  are shown in Fig. 9 [49]. It is interesting that the famous  $90^\circ$  scattering is observed as in the BPS monopole [50] and  $CP(1)$  solitons [51]. However, as can be seen in the energy level shown in Fig. 10, the solutions have higher energies for larger  $a$ , which means the moduli motion is not adiabatic since energy costs to separate the two solitons. This is due to the fact that the chiral solitons does not saturate the so called Bogomol'nyi bound. Thus if the two solitons are initially far apart, they will coalesce to form a bound state, forming a torus shape but no scattering.

## 4 Symmetry and the Degeneracy of the Quarks

The bunch of valence spectra due to the potential with discrete symmetries has been observed in the study of heavier nuclear systems. In Ref. [52], the valence spectra are highly degenerate because the deformation of the spherically symmetric shell produces large shell gaps. Thus the nuclei can be considered to be more stable than the spherical one. As discussed in Ref. [18], the group theory should predict the

level structure of pion fluctuations. However, our problem is more complicated due to the presence of quarks. Before discussing it in detail, let us show how the shell deformation is related to the degeneracy of the spectrum.

In general, if an eigenequation given by

$$H\psi_\mu = E_\mu\psi_\mu \quad (52)$$

is invariant under a symmetric operation  $\hat{R} \in \hat{g}$ , the equation transforms as

$$\hat{R}H\psi_\mu = H(\hat{R}\psi_\mu) = E_\mu\psi_\mu. \quad (53)$$

Therefore the states  $\{\psi_\mu, \hat{R}\psi_\mu\}$  are degenerate in energy with  $E_\mu$ . The set of  $d_\mu$  eigenfunctions  $\{\psi_i^{(\mu)}\} (i = 1, \dots, d_\mu)$  belonging to a given eigenvalue  $E_\mu$  will provide the basis for an irreducible representation of the group  $\hat{g}$  of the hamiltonian [53]:

$$\hat{R}\psi_j^{(\mu)} = \sum_i \psi_i^{(\mu)} D_{ij}^{(\mu)}(\hat{R}). \quad (54)$$

The Dirac equation

$$(i\gamma^\mu\partial_\mu - M)\psi(x) = 0 \quad (55)$$

is a wave equation for fermions with Lorentz-covariance, i.e. its form has to be invariant under a transition from one inertial system to another one. Let the coordinates of the event be  $x^\mu$  for an observer A and  $x'^\mu$  for an observer B. Both coordinates are connected by the Lorentz transformation

$$x' = ax \quad \text{or} \quad (x')^\nu = a_\mu^\nu x^\mu. \quad (56)$$

The equation are invariant under this transformation as

$$(i\gamma^\mu\partial_\mu - M)\psi(x) = 0 \quad \Leftrightarrow \quad (i\gamma'^\mu\partial'_\mu - M)\psi'(x') = 0. \quad (57)$$

The Dirac fields are hence transformed via

$$\psi'(x') = \psi'(\hat{a}x) \equiv \hat{S}(\hat{a})\psi(x) = \hat{S}(\hat{a})\psi(\hat{a}^{-1}x') \quad (58)$$

and also

$$\begin{aligned} \psi(x) &= \hat{S}^{-1}(\hat{a})\psi'(x') = \hat{S}^{-1}(\hat{a})\psi'(\hat{a}x), \\ \psi'(x') &= \hat{S}(\hat{a})\psi'(\hat{a}^{-1}x') \Rightarrow \psi(x) = \hat{S}(\hat{a}^{-1})\psi'(\hat{a}x). \end{aligned}$$

One can thus obtain the relation for the transformation operator  $\hat{S}(a)$  as

$$\hat{S}(\hat{a}^{-1}) = \hat{S}^{-1}(\hat{a}), \quad (59)$$



which reads to

$$(\hat{S}(\hat{a})i\gamma^\mu\partial_\mu\hat{S}^{-1}(\hat{a}) - M)\psi'(x') = 0. \quad (60)$$

In Eq. (56), performing transformation to the coordinates of the system B,

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu} = \hat{a}_\mu^\nu \frac{\partial}{\partial x'^\nu} \quad (61)$$

one gets

$$(\hat{S}(\hat{a})i\gamma^\mu\hat{S}^{-1}(\hat{a})\hat{a}_\mu^\nu\partial'_\nu - M)\psi'(x') = 0. \quad (62)$$

From the covariance of the Dirac equation in Eq. (57),  $\hat{S}(\hat{a})$  must have the following property

$$\hat{a}_\mu^\nu\gamma^\mu = \hat{S}^{-1}(\hat{a})\gamma^\nu\hat{S}(\hat{a}). \quad (63)$$

Let us consider the transformation law for the equation including the Skyrme chiral fields

$$(i\gamma^\mu\partial_\mu - MU^{\gamma_5}(x))\psi(x) = 0. \quad (64)$$

If the chiral fields have a point group symmetry  $\hat{G}$  such as

$$U(x') = \hat{G}(\hat{a})U(x)\hat{G}(\hat{a})^\dagger, \quad (\hat{G}(\hat{a}) \in SU(2)_I), \quad (65)$$

the Dirac equation is invariant under the Lorentz transformation

$$(x')^\nu = a_\mu^\nu x^\mu \quad \text{or} \quad x' = \hat{a}x \quad (66)$$

with

$$x' = \begin{pmatrix} t \\ \mathbf{x}' \end{pmatrix}, \quad \hat{a} = \begin{pmatrix} 1 & 0 \\ 0 & \hat{\mathbf{a}} \end{pmatrix}, \quad x = \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}, \quad (67)$$

accompanying a corresponding iso-rotation

$$(i\gamma^\mu\partial_\mu - MU^{\gamma_5}(x))\psi(x) = 0 \Leftrightarrow (i\gamma^\nu\partial'_\nu - MU^{\gamma_5}(x'))\psi'(x') = 0, \quad (68)$$

with

$$\psi'(x') = \hat{K}(\hat{a})\psi(x) \equiv (\hat{S}(\hat{a}) \times \hat{G}(\hat{a}))\psi(x). \quad (69)$$

The operator  $\hat{R}$  corresponding to this rotation is thus defined by

$$\psi'(x) \equiv_{def} \hat{R}\psi(x) = \hat{K}(\hat{a})\psi(\hat{a}^{-1}x). \quad (70)$$

It can be shown that the operator  $\hat{R}$  commutes with the hamiltonian using following commutations :

$$\begin{aligned}
U^{\gamma_5}(x)\psi(x) &= \hat{G}^{-1}U^{\gamma_5}(x')\hat{G} \cdot \hat{R}^{-1}\psi'(x') \\
&= \hat{G}^{-1}\hat{S}^{-1}U^{\gamma_5}(x')\psi'(x'), \\
\hat{R}\psi(x') &= \hat{K}\psi(x) \rightarrow \psi(x) = \hat{K}^{-1}\hat{R}\psi(x'), \\
\Rightarrow \hat{K}\hat{K}^{-1}\hat{R}U^{\gamma_5}(x')\psi(x') &= U^{\gamma_5}(x')\hat{R}\psi(x'), \\
\therefore [\hat{R}, U^{\gamma_5}] &= 0,
\end{aligned} \tag{71}$$

$$\begin{aligned}
i\gamma^0\gamma^k\partial_k\psi(x) &= i\gamma^0\gamma^ka_k^l\partial_l'\psi(x) \\
&= i\hat{S}^{-1}\gamma^0\gamma^l\hat{S}\partial_l'\psi(x) \\
&= i\hat{S}^{-1}\gamma^0\gamma^l\hat{S}\partial_l'\hat{K}^{-1}\psi'(x') \\
&= i\hat{G}^{-1}\hat{S}^{-1}\gamma^0\gamma^l\hat{S}\partial_l'\psi'(x') \\
\Rightarrow \hat{K}i\gamma^0\gamma^k\partial_k\psi(x) &= i\gamma^0\gamma^k\partial_k'\psi'(x') \\
\therefore [\hat{R}, i\gamma^0\gamma^k\partial_k] &= 0.
\end{aligned} \tag{72}$$

which reads  $[\hat{R}, H] = 0$ .

Generally speaking, if eigenequation

$$H\phi_k = \epsilon_k\phi_k \tag{73}$$

is invariant in the group  $G$ , in terms of the symmetric operation  $\hat{R}$  ( $\in G$ ) the equation becomes

$$\hat{R}H\phi_k = H(\hat{R}\phi_k) = \epsilon_k\phi_k. \tag{74}$$

Then,  $\phi_k$  and  $\hat{R}\phi_k$  are degenerate in energy. Thus, constructing  $\hat{R}$  for each symmetry of the hamiltonian, one should be able to deduce the degeneracy structure of the spectra occurring in the valence level.

As an example, let us examine a rotational operation for the hedgehog ansatz given in Eq. (16). The chiral fields exhibit spherical symmetry and its infinitesimal spatial rotation is of the form

$$x'^\nu = a_\mu^\nu x^\mu \rightarrow x'^\nu = x^\nu + \epsilon_\mu^\nu x^\mu. \tag{75}$$

Introducing the small angle  $\epsilon$  as

$$\epsilon_3^2 = -\epsilon_2^3 = \epsilon_1, \quad \epsilon_1^3 = -\epsilon_3^1 = \epsilon_2, \quad \epsilon_2^1 = -\epsilon_1^2 = \epsilon_3, \tag{76}$$

the Dirac field transforms as

$$\begin{aligned}
\psi(a^{-1}x) &= \psi(x^\nu - \epsilon_\mu^\nu x^\mu) \\
&= \psi(x^\nu) + \frac{\partial\psi}{\partial x^\nu}(-\epsilon_\mu^\nu) \\
&= \psi(x^\nu) + \epsilon \cdot (\mathbf{x} \times \nabla)\psi(x^\nu) \\
&= (1 + i(\mathbf{x} \times \mathbf{p}) \cdot \epsilon)\psi(x^\nu) = \exp[i\mathbf{l} \cdot \epsilon]\psi(x^\nu),
\end{aligned} \tag{77}$$

where  $\mathbf{l} = \mathbf{x} \times \mathbf{p}$  is orbital angular momentum operator. An infinitesimal transformation of the Lorentz and isorotation operators are given by

$$\begin{aligned}\hat{S} &= 1 - \frac{i}{4}\sigma^{\mu\nu}\epsilon_{\mu\nu} = 1 + \frac{i}{2}\boldsymbol{\Sigma} \cdot \boldsymbol{\epsilon}, \quad \boldsymbol{\Sigma} \equiv \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \\ \hat{I} &= 1 + \frac{i}{2}\boldsymbol{\tau} \cdot \boldsymbol{\epsilon}.\end{aligned}\tag{78}$$

Thus the transformation of the Dirac field under the Lorentz and isorotation is given by (see Eq. (70))

$$\begin{aligned}\psi'(x) = \hat{R}\psi(x) &= \hat{S} \times \hat{I}\psi(a^{-1}x) \\ &= (1 + \frac{i}{2}\boldsymbol{\Sigma} \cdot \boldsymbol{\epsilon})(1 + \frac{i}{2}\boldsymbol{\tau} \cdot \boldsymbol{\epsilon})(1 + i(\mathbf{x} \times \mathbf{p}) \cdot \boldsymbol{\epsilon})\psi(x) \\ &\sim (1 + i(\frac{1}{2}\boldsymbol{\Sigma} + \frac{1}{2}\boldsymbol{\tau} + \mathbf{x} \times \mathbf{p}) \cdot \boldsymbol{\epsilon})\psi(x) = (1 + i\mathbf{K} \cdot \boldsymbol{\epsilon})\psi(x).\end{aligned}\tag{79}$$

This  $\mathbf{K}$  is a grandspin operator and a good quantum number for the hedgehog hamiltonian.

The point group symmetry has a discrete operation so that it is not possible to perform these infinitesimal transformation analysis. Nevertheless, one can see that the rotation in Eq. (70) produces the degenerate eigenstates. Let us show briefly the derivation of the rotation operator  $\hat{R}$  for  $B = 3$  tetrahedron and the transformation law for the numerical basis constructed in Eq. (173). The  $B = 3$  tetrahedral soliton is characterized by two symmetry operations [18]:  $Z_2 \times Z_2$  and  $T_d$ . Specifically,  $Z_2 \times Z_2$  are characterized by a following Möbius transformation:

$$z \rightarrow -\frac{1}{z} \Leftrightarrow R(z) \rightarrow -\frac{1}{R(z)} \equiv R'(z)\tag{80}$$

resultantly,

$$\begin{aligned}\hat{\mathbf{n}}_R(z') &= (n_1, n_2, n_3) \\ \rightarrow \hat{\mathbf{n}}'_R(z) &= \frac{1}{1 + |R(z)|^2} (2\text{Re}[R'(z)], 2\text{Im}[R'(z)], 1 - |R'(z)|^2) \\ &= \frac{1}{1 + 1/|R|^2} (2\text{Re}[-1/R], 2\text{Im}[-1/R], 1 - |1/R|^2) \\ &= (-n_1, n_2, -n_3).\end{aligned}\tag{81}$$

The transformation operator  $\hat{g} \equiv \exp[-i\frac{\pi}{2}\tau_2]$  ensures

$$\begin{aligned}\hat{g}(\boldsymbol{\tau} \cdot \mathbf{n}_R)\hat{g}^\dagger &= (-i\tau_2)(\tau_1 n_1 + \tau_2 n_2 + \tau_3 n_3)(i\tau_2) \\ &= -n_1\tau_1 + n_2\tau_2 - n_3\tau_3 \equiv \boldsymbol{\tau} \cdot \mathbf{n}'_R.\end{aligned}\tag{82}$$

and transforms the chiral field as

$$U(\mathbf{x}') = \hat{g}U(\mathbf{x})\hat{g}^\dagger.\tag{83}$$

$T_d$  transforms the  $z$  and  $R(z)$  as

$$z \rightarrow \frac{iz + 1}{-iz + 1} \equiv z' \Leftrightarrow R(z) \rightarrow \frac{iR + 1}{-iR + 1} \equiv R'(z), \quad (84)$$

and hence

$$\begin{aligned} n_1 + in_2 &\rightarrow n'_1 + in'_2 = \frac{2R'}{1 + |R'|^2} = \frac{1 + i(R + \bar{R}) - |R|^2}{1 + |R|^2} = n_3 + in_1, \\ n_1 - in_2 &\rightarrow n'_1 - in'_2 = \frac{2\bar{R}'}{1 + |R'|^2} = \frac{1 - i(R + \bar{R}) - |R|^2}{1 + |R|^2} = n_3 - in_1, \\ n_3 &\rightarrow n'_3 = \frac{1 - |R'|^2}{1 + |R'|^2} = \frac{-i(R - \bar{R})}{1 + |R|^2} = n_2, \end{aligned} \quad (85)$$

which yields

$$\begin{aligned} n'_1\tau_1 + n'_2\tau_2 + n'_3\tau_3 &= \frac{1}{2}(n'_1 + in'_2)(\tau_1 - i\tau_2) + \frac{1}{2}(n'_1 - in'_2)(\tau_1 + i\tau_2) + n'_3\tau_3 \\ &= \frac{1}{2}(n_3 + in_1)(\tau_1 - i\tau_2) + \frac{1}{2}(n_3 - in_1)(\tau_1 + i\tau_2) + n_2\tau_3 \\ &= n_3\tau_1 + n_1\tau_2 + n_2\tau_3. \end{aligned} \quad (86)$$

The transformation operator  $\hat{h} \equiv \exp[-i\frac{\pi}{3}\frac{1}{\sqrt{3}}(\tau_1 + \tau_2 + \tau_3)] = (1 - i(\tau_1 + \tau_2 + \tau_3))/2$  ensures

$$\begin{aligned} \hat{h}(\boldsymbol{\tau} \cdot \mathbf{n}_R)\hat{h}^\dagger &= \frac{1}{2}(1 - i(\tau_1 + \tau_2 + \tau_3))(\tau_1 n_1 + \tau_2 n_2 + \tau_3 n_3)\frac{1}{2}(1 + i(\tau_1 + \tau_2 + \tau_3)) \\ &= n_3\tau_1 + n_1\tau_2 + n_2\tau_3 \equiv \boldsymbol{\tau} \cdot \mathbf{n}'_R, \end{aligned} \quad (87)$$

and transforms the chiral field as

$$U(\mathbf{x}') = \hat{h}U(\mathbf{x})\hat{h}^\dagger. \quad (88)$$

The Lorentz transformation operator  $\hat{S}$  and the operators for the chiral fields  $\{\hat{g}, \hat{h}\}$  corresponding to the symmetric operations  $(x')^\nu = a_\mu^\nu x^\mu$  are given by

$$\begin{aligned} Z_2 \times Z_2 : \\ (a_g)^\nu_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Rightarrow \hat{S}_g = i\gamma^0\gamma^5\gamma^2, \end{aligned} \quad (89)$$

$$\begin{aligned} T_d : \\ (a_h)^\nu_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \Rightarrow \hat{S}_h = \exp[i\frac{\pi}{3}\frac{1}{\sqrt{3}}(\sigma_{23} + \sigma_{31} + \sigma_{12})]. \end{aligned} \quad (90)$$

The  $\hat{R}$  is defined by the direct product of these rotation operators together with the inverse spatial rotation for the spinor such as

$$\hat{R}_g \psi(x) \equiv (\hat{S}_g \times \hat{g}) \psi(\hat{a}_g^{-1} x), \quad (91)$$

$$\hat{R}_h \psi(x) \equiv (\hat{S}_h \times \hat{h}) \psi(\hat{a}_h^{-1} x). \quad (92)$$

Applying these operators to the Kahana-Ripka basis  $\phi \equiv \{u, v\}$  (for detail, see Appendix B) we finally obtain the following transformation laws:

$$\hat{R}_g \phi_{KM} = (-1)^{K-M} \phi_{K-M}. \quad (93)$$

$$\hat{R}_h \phi_{00} = \phi_{00}. \quad (94)$$

$$\hat{R}_h \begin{pmatrix} \phi_a \\ \phi_b \\ \phi_c \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_a \\ \phi_b \\ \phi_c \end{pmatrix}, \quad (95)$$

$$\phi_a \equiv \frac{1}{\sqrt{2}}(\phi_{11} + \phi_{1-1}), \quad \phi_b \equiv i\phi_{10}, \quad \phi_c \equiv \frac{i}{\sqrt{2}}(\phi_{11} - \phi_{1-1}).$$

$$\hat{R}_h \begin{pmatrix} \phi_\xi \\ \phi_\eta \\ \phi_\zeta \\ \phi_u \\ \phi_v \end{pmatrix} = \left( \begin{array}{ccc|cc} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{array} \right) \begin{pmatrix} \phi_\xi \\ \phi_\eta \\ \phi_\zeta \\ \phi_u \\ \phi_v \end{pmatrix}, \quad (96)$$

$$\phi_\xi \equiv \frac{i}{\sqrt{2}}(\phi_{21} + \phi_{2-1}), \quad \phi_\eta \equiv \frac{1}{\sqrt{2}}(\phi_{21} - \phi_{2-1}),$$

$$\phi_\zeta \equiv \frac{i}{\sqrt{2}}(\phi_{22} - \phi_{2-2}), \quad \phi_u \equiv \phi_{20}, \quad \phi_v \equiv \frac{1}{\sqrt{2}}(\phi_{22} + \phi_{2-2}),$$

confirming that the  $B = 3$  tetrahedron can exhibit triply degenerate spectra.

The numerical computation indicates that the winding number strongly couple the elements with different  $K$  and correlated valence spectra occur as a result. As can be seen from the operator  $K$  with  $B = 2$ , the degeneracy of the valence spectra are explained in terms of the shape deformation (symmetry) as well as the winding number of the chiral fields [13]. The four-fold degeneracy of the lowest states may be ascribed to the chiral symmetry  $SU(2)_L \times SU(2)_R$  of the hamiltonian. The degenerate structure for  $B \geq 3$  will be well understood if symmetric operators of the hamiltonian which consist of the angular momentum, spin, isospin and winding number, are explicitly constructed.

## 5 Zero Mode Quantization

The solitons that we obtained in the previous section are classical objects and therefore must be quantized to assign definite spin and isospin to them. Quantization of

the solitons can be performed semiclassically for their rotational zero modes. For the hedgehog soliton, because of its topological structure, a rotation in isospin space is followed by a simultaneous spatial rotation. For the axially symmetric soliton, there are five rotational zero modes by rotations of iso-degrees of freedom and spatial rotations.

### 5.1 $SU(2)$ sector

Let us introduce the dynamically rotated chiral fields around the classical fields  $U_S$  [37]:

$$U(\mathbf{x}, t) = A(t)U_S(\mathbf{x}')A(t)^\dagger, \quad x^{i'} = \Xi_j^i[B(t)]x^j \quad (97)$$

where

$$\Xi_j^i[B(t)] = \frac{1}{2} \text{Tr}[\sigma^i B(t) \sigma_j B(t)^\dagger], \quad (98)$$

and  $A(t)$  and  $B(t)$  are time-dependent  $SU(2)$  matrices generating an iso-rotation and a spatial rotation respectively. By transforming the rotating frame of reference, the Dirac operator with Eq. (97) can be written as

$$\begin{aligned} i\tilde{D} &= i\cancel{\partial} - MU^{\gamma^5}(\mathbf{x}, t) \\ &= A(t)S(t)^\dagger \gamma^0 [i\partial_t + i\tilde{\gamma}^0 \tilde{\gamma}^k \partial_k - MU_S^{\gamma^5}(\mathbf{x}') + iA^\dagger \dot{A} + iS^\dagger \dot{S}] S(t)A(t)^\dagger \end{aligned} \quad (99)$$

where

$$\tilde{\gamma}^\mu = \Lambda_\nu^\mu S \gamma^\nu S^\dagger = \begin{pmatrix} \gamma^0 \\ \gamma^k + (\mathbf{r}' \times \dot{\boldsymbol{\theta}})^k \gamma^0 \end{pmatrix}, \quad (100)$$

and  $S(t)$  is the rotation operator for the Dirac field and  $\boldsymbol{\theta}$  is an angle of the spatial rotation. Note that the gamma matrices  $\tilde{\gamma}^\mu$  explicitly depend on the coordinates and do not transform as a contravariant vector [54]. Substituting Eq. (100) into Eq. (99), one obtains

$$i\tilde{D} = A(t)S(t)^\dagger \gamma^0 [i\partial_t - H(U_S^{\gamma^5}) + \Omega_A + \Omega_B] S(t)A(t)^\dagger \quad (101)$$

where

$$\Omega_A = iA^\dagger \dot{A} = \frac{1}{2} \Omega_A^a \tau_a, \quad \Omega_B = iS^\dagger \dot{S} + (\mathbf{r} \times \mathbf{p}) \cdot \dot{\boldsymbol{\theta}} = \Omega_B^a J_a \quad (102)$$

with  $J_a = 1/2 \epsilon_{abc} \gamma^b \gamma^c - i(\mathbf{r} \times \nabla)_a$ .  $\Omega_A$  and  $\Omega_B$  are the angular velocity operators for an isorotation and for a spatial rotation respectively. Under these operations the effective action can be written by

$$\begin{aligned} S_{\text{eff}}(U) &= S_{\text{eff}}(U_S) \\ &- iN_c \text{Sp} [\log(i\partial_t - H(U_S^{\gamma^5}) + \Omega_A + \Omega_B)] - \text{Sp} [\log(i\partial_t - H(U_S^{\gamma^5}))] \\ &\rightarrow S_{\text{eff}}(U_S) - \frac{1}{2} iN_c \text{Sp} \log dd^\dagger + \frac{1}{2} iN_c \text{Sp} \log d_0 d_0^\dagger. \end{aligned} \quad (103)$$

With the proper-time regularization, it reads

$$S_{\text{eff}}^{\text{reg}}(U) = S_{\text{eff}}^{\text{reg}}(U_S) - \frac{N_c}{2} \int \frac{d\omega}{2\pi} \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \text{Sp}[e^{-dd^\dagger\tau} - e^{-d_0d_0^\dagger\tau}], \quad (104)$$

where

$$d = i\omega - H(U_S^{\gamma_5}) + \Omega_A + \Omega_B, d^\dagger = -i\omega - H(U_S^{\gamma_5}) - \Omega_A - \Omega_B, \quad (105)$$

$$dd^\dagger = \omega^2 + H^2 - 2i\omega(\Omega_A + \Omega_B) - [H, \Omega_A + \Omega_B] - (\Omega_A + \Omega_B)^2, \quad (106)$$

$$d_0d_0^\dagger = \omega^2 + H^2. \quad (107)$$

Assuming that the rotation of the soliton is adiabatic, we shall expand the effective action  $S_{\text{eff}}$  around the classical solution  $U_S$  with respect to the angular momentum velocity  $\Omega_A$  and  $\Omega_B$  up to second order [55]

$$\begin{aligned} S_{\text{eff}}^{\text{reg}}(U) &= S_{\text{eff}}^{\text{reg}}(U_S) \\ &+ \frac{1}{2} \sum_{ab} \int dt [I_{0,ab}^{AA} \Omega_A^a(t) \Omega_A^b(t) + I_{0,ab}^{AB} \Omega_A^a(t) \Omega_B^b(t) \\ &+ I_{0,ab}^{BA} \Omega_B^a(t) \Omega_A^b(t) + I_{0,ab}^{BB} \Omega_B^a(t) \Omega_B^b(t)] \end{aligned} \quad (108)$$

where  $I_{0,ab}$ 's are the vacuum sea contributions to the moments of inertia defined by

$$\begin{aligned} I_{0,ab}^{AA} &= \frac{1}{8} N_c \sum_{n,m} f(E_m, E_n, \Lambda) \langle n | \tau_a | m \rangle \langle m | \tau_b | n \rangle, \\ I_{0,ab}^{AB} &= \frac{1}{4} N_c \sum_{n,m} f(E_m, E_n, \Lambda) \langle n | \tau_a | m \rangle \langle m | J_b | n \rangle, \\ I_{0,ab}^{BA} &= \frac{1}{4} N_c \sum_{n,m} f(E_m, E_n, \Lambda) \langle n | J_a | m \rangle \langle m | \tau_b | n \rangle, \\ I_{0,ab}^{BB} &= \frac{1}{2} N_c \sum_{n,m} f(E_m, E_n, \Lambda) \langle n | J_a | m \rangle \langle m | J_b | n \rangle \end{aligned}$$

with the cutoff function  $f(E_m, E_n, \Lambda)$

$$\begin{aligned} f(E_m, E_n, \Lambda) &= -\frac{2\Lambda}{\sqrt{\pi}} \frac{e^{-E_m^2/\Lambda^2} - e^{-E_n^2/\Lambda^2}}{E_m^2 - E_n^2} \\ &+ \frac{\text{sgn}(E_m) \text{erfc}(|E_m|/\Lambda) - \text{sgn}(E_n) \text{erfc}(|E_n|/\Lambda)}{E_m - E_n}. \end{aligned} \quad (109)$$

Similarly, for the valence quark contribution to the moments of inertia, we have

$$\begin{aligned}
I_{\text{val},ab}^{AA} &= \frac{1}{2} N_c \sum_{m \neq \text{val}} \frac{\langle \text{val} | \tau_a | m \rangle \langle m | \tau_b | \text{val} \rangle}{E_m - E_{\text{val}}}, \\
I_{\text{val},ab}^{AB} &= N_c \sum_{m \neq \text{val}} \frac{\langle \text{val} | \tau_a | m \rangle \langle m | J_b | \text{val} \rangle}{E_m - E_{\text{val}}}, \\
I_{\text{val},ab}^{BA} &= N_c \sum_{m \neq \text{val}} \frac{\langle \text{val} | J_a | m \rangle \langle m | \tau_b | \text{val} \rangle}{E_m - E_{\text{val}}}, \\
I_{\text{val},ab}^{BB} &= 2N_c \sum_{m \neq \text{val}} \frac{\langle \text{val} | J_a | m \rangle \langle m | J_b | \text{val} \rangle}{E_m - E_{\text{val}}}.
\end{aligned} \tag{110}$$

The total moments of inertia are then given by the sum of the vacuum and valence as  $I_{ab}^{AA} = I_{\text{val},ab}^{AA} + I_{0,ab}^{AA}$ .

Finally, the effective lagrangian is obtained as

$$L = -E_{\text{static}} + \frac{1}{2} I_{ab}^{AA} \Omega_A^a \Omega_A^b + I_{ab}^{AB} \Omega_A^a \Omega_B^b + \frac{1}{2} I_{ab}^{BB} \Omega_B^a \Omega_B^b. \tag{111}$$

Theoretically, these moments of inertia can be computed using the eigenstates of Eq. (8). However, due to the difference of the boundary conditions between the initial and final states of the matrix element, the moments of inertia acquire nonzero values with vanishing pion fields. To overcome this problem, we make the following replacement [56]:

$$\begin{aligned}
\langle n | J_a | m \rangle &\rightarrow \langle n | [H(U_S^{\gamma_5}), J_a] | m \rangle / (E_n - E_m) \\
&= \langle n | [MU_S^{\gamma_5}, l_a] | m \rangle / (E_n - E_m)
\end{aligned} \tag{112}$$

where  $l_a = -i(\mathbf{r} \times \nabla)_a$ . Unless the hamiltonian explicitly depend on the coordinates, the numerator vanishes with vanishing pion fields. The spurious contributions to the moment of inertia can be removed in this way.

From axial symmetry of the system, following relations are derived

$$\begin{aligned}
I_{ij} &= 0, \quad i \neq j; \quad I_{11}^{AB} = I_{22}^{AB} = I_{11}^{BA} = I_{22}^{BA} = 0, \\
I_{11}^{AA} &= I_{22}^{AA}, \quad I_{11}^{BB} = I_{22}^{BB}, \quad I_{33}^{BB} = m^2 I_{33}^{AA}, \quad I_{33}^{AB} = I_{33}^{BA} = -m_w I_{33}^{AA}.
\end{aligned} \tag{113}$$

The quantization conditions for the collective coordinates,  $A(t)$  and  $B(t)$ , define a body-fixed isospin operator  $\mathbf{K}$  and a body-fixed angular momentum operator  $\mathbf{L}$  as

$$I_{ab}^{AA} \Omega_A^b + I_{ab}^{AB} \Omega_B^b \rightarrow -\text{tr} \left( A \frac{\tau_a}{2} \frac{\partial}{\partial A} \right) \equiv -K_a, \tag{114}$$

$$I_{ab}^{BA} \Omega_A^b + I_{ab}^{BB} \Omega_B^b \rightarrow \text{tr} \left( \frac{\sigma_a}{2} B \frac{\partial}{\partial B} \right) \equiv -L_a. \tag{115}$$



Table 3: Moments of inertia (in  $\text{MeV}^{-1}$ ).

$B$		Valence	Sea	Total	$B$		Valence	Sea	Total
2	$I_{11}^{AA}$	0.00773	0.00363	0.01136	4	$I_{11}^{AA}$	0.01408	0.00959	0.02366
	$I_{11}^{BB}$	0.01141	0.00464	0.01605		$I_{11}^{BB}$	0.04272	0.01245	0.05517
	$I_{33}^{AA}$	0.00429	0.00125	0.00554		$I_{33}^{AA}$	0.01172	0.00074	0.01246
3	$I_{11}^{AA}$	0.01231	0.00280	0.01511	5	$I_{11}^{AA}$	0.02786	0.00716	0.03502
	$I_{11}^{BB}$	0.02174	0.00384	0.02558		$I_{11}^{BB}$	0.12124	0.01112	0.13236
	$I_{33}^{AA}$	0.00594	0.00027	0.00622		$I_{33}^{AA}$	0.01368	0.00007	0.01375

These are related to the usual coordinate-fixed isospin operator  $I_a$  and coordinate-fixed angular momentum  $J_a$  operator by transformations,

$$I_a = -\Xi_a^b[A(t)]K_b, \quad J_a = -\Xi_a^b[B(t)]^T L_b. \quad (116)$$

To estimate the quantum energy corrections, let us introduce the basis functions of the spin and isospin operators which were inspired from the cranking method for nuclei [39, 36],

$$\langle A, B | i i_3 k_3, j j_3 l_3 \rangle = \frac{\sqrt{(2i+1)(2j+1)}}{8\pi^2} D_{i_3 k_3}^i(A) D_{j_3, -m_w k_3}^j(B)$$

where  $D$  is the Wigner rotation matrix. Then, we find the quantized energies of the soliton as

$$E = E_{\text{static}} + \frac{1}{2I_{11}^{AA}} i(i+1) + \frac{1}{2I_{11}^{BB}} j(j+1) + \frac{1}{2} \left( \frac{1}{I_{33}^{AA}} - \frac{1}{I_{11}^{AA}} - \frac{m_w^2}{I_{11}^{BB}} \right) k_3^2 \quad (117)$$

where  $i(i+1)$ ,  $j(j+1)$  and  $k_3$  are eigenvalues of the Casimir operators  $\mathbf{I}^2$  and  $\mathbf{J}^2$ , and the operator  $\mathbf{K}_3$ , respectively.

In Table 3 are the results of our calculation of moments of inertia,  $I_{11}^{AA}$ ,  $I_{11}^{BB}$  and  $I_{33}^{AA}$ , with  $B = 2 - 5$ . It is instructive to compare our results with the Skyrme model [37] where  $U_{11} = 0.0104$ ,  $V_{11} = 0.0163$  and  $U_{33} = 0.00709$  which are correspondingly our  $I_{11}^{AA}$ ,  $I_{11}^{BB}$  and  $I_{33}^{AA}$ . They are qualitatively in good agreement.

## 5.2 Finkelstein-Rubinstein constraints

If a multi-skyrmion describes atomic nuclei upon quantization, it has to be quantized as a boson or as a fermion whether  $B$  is even or odd. This requirement is implemented in the form of Finkelstein-Rubinstein (FR) constraints [57]. For highly nonlinear theory enough to possess soliton solutions a consideration of continuity reduces to a concept of distinct topological sector. Finkelstein and Rubinstein state that an equally primitive concept as continuity is between multi-valued quantized

systems which can possess state functions double-valued under  $2\pi$  rotation, and those which cannot. Both concepts are ascribed to homotopy of the map between the physical space and the configuration space. Indeed for the  $SU(2)$  chiral soliton solution to exist, the physical space must be compactified to  $S^3$  which defines a topological charge characterized by an integer

$$\pi_3(S^3) = n. \quad (118)$$

The FR constraints arise when the space-time is suitably compactified as

$$\pi_4(SU(2)) = Z_2, \quad (119)$$

which takes values only  $-1$  or  $+1$ . This allows us to quantize the solitons as either a fermion or a boson.

The FR constraints for the rational map ansatz was constructed in Ref. [22] and Ref. [58] and applied to predict the ground states of skyrmions up to  $B = 22$ . In this section, we shall apply the FR constraints for the rational map ansatz directly to our axially symmetric multi-skyrmions and obtain their ground states.

Following the notation in Ref. [58], let  $g$  be a rotation by  $\alpha$  around  $\mathbf{n}$  followed by an isorotation by  $\beta$  around  $\mathbf{N}$ . Then the FR constraints can be defined by

$$\exp(-i\alpha\mathbf{n} \cdot \mathbf{J}) \exp(-i\beta\mathbf{N} \cdot \mathbf{I})\psi = \chi_{FR}(g)\psi \quad (120)$$

where

$$\chi_{FR}(g) = \begin{cases} 1 & \text{if contractible} \\ -1 & \text{otherwise.} \end{cases} \quad (121)$$

and,  $\mathbf{J}$  and  $\mathbf{I}$  are space-fixed spin and isospin operators respectively.  $\psi$  is the wave function which transforms under a tensor product of rotations and isorotations. In particular, a closed loop is noncontractible for odd  $B$  and contractible for even  $B$ , which is consistent with spin statistics. Consequently, quantum numbers  $I$  and  $J$  are half-integers for odd  $B$  and integers for even  $B$ .

In order to construct the ground states for a given baryon number  $B$ , let us define  $N(L(\alpha, \beta))$  as a homotopy invariant for a loop  $L$  generated by rotations by  $\alpha$  and isorotations by  $\beta$ . Then, for the axially symmetric rational map of degree  $B$ , it is given by Ref. [58]

$$N(L(\alpha, \beta)) = \frac{B}{2\pi}(B\alpha - \beta). \quad (122)$$

It can be shown that  $N \pmod{2}$  determines if the loop is contractible or not in the same sense as  $B \pmod{2}$ . Therefore,  $N \pmod{2}$  gives the FR constraints for each generator of the symmetry group of the rational map.

The axially symmetric rational map with degree  $B$  is given by

$$R(z) = \frac{1}{z^B}. \quad (123)$$

There are two symmetric generators for this rational map. One is a rotation by  $\alpha$  followed by an isorotation by  $\beta = B\alpha$ . Substituting it into Eq. (122), one obtains  $N(L(\alpha, B\alpha)) = 0$ . The FR constraints for this loop is thus given by

$$e^{-i\pi(L_3 - BK_3)}\psi = \psi. \quad (124)$$

where  $L_3$ ,  $K_3$  are the third component of the body-fixed angular momentum and the isospin operators which are related with the space-fixed operators by orthogonal transformations (116). The other symmetry is  $C_2$  with transformation

$$z \rightarrow \frac{1}{z}, \quad R(z) \rightarrow \frac{1}{R(z)}. \quad (125)$$

This corresponds to  $\alpha = \beta = \pi$  and hence  $N(L(\pi, \pi)) = B(B-1)/2$ . The FR constraints for this loop is

$$e^{-i\pi(L_1 + K_1)}\psi = (-1)^{B(B-1)/2}\psi. \quad (126)$$

In the following we construct the ground states consistent with the derived FR constraints (124) and (126) for  $B = 2 - 5$  with axial symmetry.

- $B = 2$

We find the FR constraints

$$e^{-i\pi(L_3 - 2K_3)}\psi = \psi \quad (127)$$

$$e^{-i\pi(L_1 + K_1)}\psi = -\psi. \quad (128)$$

This gives the ground state as  $|J, L_3\rangle |I, K_3\rangle = |1, 0\rangle |0, 0\rangle$ . This is in agreement with the ground state  ${}^1_1\text{H}^+$  (deuteron).

- $B = 3$

We find the FR constraints

$$e^{-i\pi(L_3 - 3K_3)}\psi = \psi \quad (129)$$

$$e^{-i\pi(L_1 + K_1)}\psi = -\psi \quad (130)$$

This gives the ground state as  $|\frac{5}{2}, \frac{3}{2}\rangle |\frac{1}{2}, \frac{1}{2}\rangle$ . The ground state (and first “excited” state) of  $B = 3$  have  $(I, J) = (\frac{1}{2}, \frac{1}{2})$  ( ${}^3_2\text{He}^+$ ,  ${}^3_1\text{H}^+$ ). Then, one can not identify our soliton with these observed isodoublet nuclei.

- $B = 4$

We find the FR constraints

$$e^{-i\pi(L_3 - 4K_3)}\psi = \psi \quad (131)$$

$$e^{-i\pi(L_1 + K_1)}\psi = \psi. \quad (132)$$

Table 4:  $B = 2$ , mass spectrum up to  $i, j \leq 3, k_3 \leq 1$ .

Classification	$(i, j, k_3)$	Parity	Mass[MeV]
$NN(^3S_1)$	(0, 1, 0)	+	2264
$NN(^1S_0)$	(1, 0, 0)	+	2290
$N\Delta(^3P_2)$	(1, 2, 1)	−	2399
$N\Delta(^5S_2)$	(1, 2, 0)	+	2477
$N\Delta(^3S_1)$	(2, 1, 0)	+	2528
$\Delta\Delta(^7S_3)$	(0, 3, 0)	+	2576
$\Delta\Delta(^1S_0)$	(3, 0, 0)	+	2730
$\Delta\Delta(^5P_3)$	(2, 3, 1)	−	2762

This gives the ground state as  $|0, 0\rangle |0, 0\rangle$ . The ground state  ${}^4_2\text{He}^+$  has  $(I, J) = (0, 0)$ . Then our soliton can be identified as the “ $\alpha$  particle”.

- $B = 5$

We find the FR constraints

$$e^{-i\pi(L_3-5K_3)}\psi = \psi \quad (133)$$

$$e^{-i\pi(L_1+K_1)}\psi = \psi. \quad (134)$$

This gives the ground state as  $|J\rangle |I\rangle = \left|\frac{7}{2}, \frac{5}{2}\right\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle$ . This is not in agreement with the observed nuclei of  $B = 5$ ;  $(I, J) = (\frac{1}{2}, \frac{3}{2})$  ( ${}^5_2\text{He}^+$ ,  ${}^5_3\text{Li}^+$ ).

Thus, we conclude that for even  $B$ , the axially symmetric solitons are possible candidates of the ground states of  $B$  atomic nuclei as is the case of the deuteron and  ${}^4_2\text{He}$  while for odd  $B$  they emerge only as excited states.

The results of the quantized energy of the axially symmetric soliton solutions with  $B = 2, 4$  are shown in Table 4 and 5. The study of the Finkelstein-Rubinstein constraints indicates that the axially symmetric solution with even  $B$  has the same quantum number as the physically observed nuclei. Some of the states may be observed in experiments. Specifically, in the  $B = 2$ , we obtained the  $I = 0, J = 1$  ( ${}^3S_1$ : deuteron) and  $I = 1, J = 0$  ( ${}^1S_0$ ) solutions. The energy of  ${}^3S_1$  is lower than the  ${}^1S_0$  because  $I_{11}^{BB} > I_{11}^{AA}$  (see Table 3). The order is in agreement with the experimental observations. For  $B = 4$ , the quantum number of the ground state  $I = 0, J = 0$  coincide with the observation. In experiment, the lowest excited state also has  $I = 0, J = 0$ , which unfortunately can not be explained within our scheme. This state can be interpreted as  ${}^3_2\text{He}+n$  bound state or  ${}^3_1\text{H}+p$  resonance state rather than the resonance of single  ${}^4_2\text{He}$ . Our formulation for the multi-soliton is based on the single bag-like picture. We expect that for some resonance states one needs advanced formulation including multi-fragments such as  $B = (3 + 1), (2 + 2)$ . We

Table 5:  $B = 4$ , mass spectrum up to  $i \leq 3, j \leq 5, k_3 \leq 1$ .

Classification	$(i, j, k_3)$	Parity	Mass[MeV]
${}^4N({}^1S_0)$	(0, 0, 0)	+	4753
${}^4N({}^5S_2)$	(0, 2, 0)	+	4807
${}^2N\,{}^2\Delta({}^7P_4)$	(0, 4, 1)	−	4808
${}^4N({}^3S_1)$	(1, 1, 0)	+	4813
${}^4N({}^1S_0)$	(2, 0, 0)	+	4879
${}^3N\,\Delta({}^7S_3)$	(1, 3, 0)	+	4904
${}^4N({}^5S_2)$	(2, 2, 0)	+	4934. <sub>2</sub>
${}^2N\,{}^2\Delta({}^7S_4)$	(0, 4, 0)	+	4934. <sub>3</sub>
${}^2N\,{}^2\Delta({}^9P_4)$	(2, 4, 1)	−	4935
$N\,{}^3\Delta({}^9P_5)$	(1, 5, 1)	−	4941
${}^3N\,\Delta({}^3S_1)$	(3, 1, 0)	+	5025
$N\,{}^3\Delta({}^9S_5)$	(1, 5, 0)	+	5046
${}^2N\,{}^2\Delta({}^9S_4)$	(2, 4, 0)	+	5061
${}^3N\,\Delta({}^7S_3)$	(3, 3, 0)	+	5115
$N\,{}^3\Delta({}^9P_5)$	(3, 5, 1)	−	5152
$N\,{}^3\Delta({}^9S_5)$	(3, 5, 0)	+	5278

observed  $I = 0, J = 2$  with positive parity as a first excited state and this channel should emerge as a higher resonance (roughly 28 MeV from the ground state) in experiment. For odd  $B$ , the constraint of  $C_2$  in Eq. (126) seems to assure the validity of the ansatz. Indeed, it provides the ground state as  $I = J = 1/2$  for  $B = 3$  and as  $I = 1/2, J = 3/2$  for  $B = 5$ , which coincide with physical observations. This seems to make sense since in the minimal energy configurations with discrete symmetries, the solutions tend to have  $I = J = 1/2$  due to their shell-like structure. However, unfortunately the constraint in Eq. (124) forbids such states. Consequently, the axially symmetric solitons with odd  $B$  emerge only as excited states. The resultant lowest state is  $E = 3657$  MeV with  $I = 1/2, J = 5/2$  for  $B = 3$ , and  $E = 6591$  MeV with  $I = 1/2, J = 7/2$  for  $B = 5$ . Experimentally, no possible candidate of the state  $I = 1/2, J = 7/2$  are found.

As stated in Sec. 2, in the soliton approach the absolute mass always tends to be higher due to the lack of the Casimir effects. Therefore the total energies is overestimated around  $0.5 \sim 1$  GeV in our calculation. The one-loop corrections and vacuum effects should be properly subtracted in order to estimate physical mass of the solutions. For  $B = 1$ , the obtained mass of the nucleon the delta is  $E_N = 1260$  MeV and  $E_\Delta = 1505$  MeV respectively because of the lack of Casimir effects. With these values, the ground states of the  $B = 2 \sim 4$  exhibit bound states but the ground state of the  $B = 5$  does not. For  $B = 2$ , the state  $I = 0, J = 3$  ( ${}^7S_3$ ) has a large binding energy. The search of this resonance is interesting subject and also

a thorough analysis of the Casimir effects for the toroidal solitons are much desired for a stability argument.

### 5.3 $SU(3)$ sector

The  $SU(2)_L \times SU(2)_R$ -invariant lagrangian has a natural extension to  $SU(3)$  sector including strange quarks. The  $SU(3)_L \times SU(3)_R$ -invariant lagrangian in the chiral quark soliton model is given by

$$\mathcal{L} = \bar{\psi}(i\bar{\partial} - MU^{\gamma_5} - \hat{m})\psi \quad (135)$$

where  $U^{\gamma_5}(x) = e^{i\gamma_5\pi_a(x)\lambda_a/f_\pi}$  and  $\lambda_a$  are the usual Gell-Mann matrices with  $\lambda_0 = \sqrt{\frac{2}{3}}\hat{1}$ . In order to estimate the effects of the symmetry breaking of  $SU(3)$  explicitly, we introduce the current quark mass matrix

$$\hat{m} = \text{diag}(m_0, m_0, m_s) = m_0\hat{1} + \Delta m(1 - \sqrt{3}\lambda_8)/3 \quad (136)$$

where  $\Delta m \equiv m_s - m_0$  is the mass difference between the strange- and the u,d-quark.

Within the proper-time regularization scheme, we fix our parameters by the input parameters, pion mass  $m_\pi$ , kion mass  $m_K$ , and the pion decay constant  $f_\pi$  in the following formulae [59, 60]

$$\frac{N_c M^2}{4\pi^2} \int_0^\infty \frac{d\tau}{\tau} \phi(\tau) e^{-\tau\bar{M}^2} = f_\pi^2 \quad (137)$$

$$m_0 \frac{N_c M}{2\pi^2 f_\pi^2} \int_0^\infty \frac{d\tau}{\tau^2} \phi(\tau) e^{-\tau\bar{M}^2} = m_\pi^2 \quad (138)$$

$$(m_0 + \frac{\Delta m}{2}) \frac{N_c M}{2\pi^2 f_\pi^2} \int_0^\infty \frac{d\tau}{\tau^2} \phi(\tau) e^{-\tau\bar{M}^2} = m_K^2 \quad (139)$$

where  $\bar{M} = M + m_0$ . For the damping function  $\phi(\tau)$  we introduce the two cut-off  $\Lambda_1, \Lambda_2$  and the parameter  $c$  via

$$\phi(\tau) = c\theta(\tau - 1/\Lambda_1^2) + (1 - c)\theta(\tau - 1/\Lambda_2^2). \quad (140)$$

Using the values  $M = 400$  MeV,  $m_0 = 6$  MeV,  $f_\pi = 93$  MeV,  $m_\pi = 138$  MeV and Eqs. (137),(138) we obtain the parameter set  $\{c = 0.76, \Lambda_1 = 433.8 \text{ MeV}, \Lambda_2 = 1512.4 \text{ MeV}\}$ . By  $m_K = 496$  MeV and Eq. (139), we determine  $m_s = 149$  MeV.

For the extension of the three flavor soliton with  $B = 2$ , we follow the usual collective coordinate approach for the hedgehog ansatz with  $B = 1$  [61, 59]. Within the collective coordinate approach, the extension to  $SU(3)$  is performed by trivial embedding [62]:

$$U(\mathbf{x}, t) = A(t) \begin{pmatrix} U_0(\Lambda_j^i(t)x^j) & 0 \\ 0 & 1 \end{pmatrix} A^\dagger(t), \quad (141)$$

where  $A(t)$  is the time-dependent  $SU(3)$  collective rotation matrix and  $\Lambda_j^i(t) = \frac{1}{2}\text{Tr}(\tau^i B \tau_j B^\dagger)$  is the spatial rotation matrix. Substituting (141) into the quark determinant and transforming the rotated frame of reference, one obtains [15]:

$$\begin{aligned} iD &= i\cancel{D} - MU^{\gamma_5}(\mathbf{x}, t) - \hat{m} \\ &\rightarrow A(t)S(t)^\dagger \gamma_0 (i\partial_t - H(U_S^{\gamma_5}) - H_{SB} - \Omega_A + \Omega_B)S(t)A(t)^\dagger, \end{aligned} \quad (142)$$

where

$$\Omega_A = -iA^\dagger \dot{A} = \frac{1}{2}\Omega_A^a \lambda_a, \quad (143)$$

$$\Omega_B = \Omega_B^a \left( \frac{1}{2}\epsilon_{abc}\gamma^b \gamma^c - i(\mathbf{r} \times \nabla)_a \right) = \Omega_B^a J_a, \quad (144)$$

and

$$H_{SB} = A^\dagger(t)\beta\Delta m \frac{1}{3}(1 - \sqrt{3}\lambda_8)A(t). \quad (145)$$

$S(t)$  is the rotation operator for the Dirac fields. In the rotating system, the quarks feel the induced Coriolis forces  $\Omega_A, \Omega_B$  and  $H_{SB}$ .  $\Omega_A, \Omega_B$  are the angular velocity operators for the right flavor rotation and the spatial rotation. The  $H_{SB}$  represents the contribution to the hamiltonian due to the  $SU(3)$  symmetry breaking.

We assume that the rotational velocities and the mass difference are relatively small and the expansion in powers of  $\Omega_A, \Omega_B$  and  $\Delta m$  is rapidly convergent. Expanding the quark determinant and the (valence quark) Green function in terms of the collective angular velocity up to the second order and the quark mass difference of the first order. The effective action (5) with Eq. (142) can be rewrite as

$$S_{\text{eff}} = -iN_c \text{Sp} \log iD \rightarrow S_{\text{eff}}(U_S) + S_R + S_I. \quad (146)$$

$S_R, S_I$  mean the real and imaginary contribution to the action, which are explicitly written as

$$S_R = -\frac{i}{2}N_c [\text{Sp} \log dd^\dagger - \text{Sp} \log d_0 d_0^\dagger], \quad S_I = -\frac{i}{2}N_c \text{Sp} \log d(d^\dagger)^{-1}, \quad (147)$$

where  $d = i\partial_t - H(U_S) - H_{SB} - \Omega_A + \Omega_B$ . The expansion for the real part with proper-time regularization form yields

$$\begin{aligned} S_R &\rightarrow -\int \frac{d\tau}{\tau} \text{Sp} \phi(\tau) [e^{-d d^\dagger \tau} - e^{-d_0 d_0^\dagger \tau}] \\ &\rightarrow \int dt \left[ -\frac{1}{2}\gamma_0(1 - D_{88}) + \frac{1}{2}I_{ab,0}^{AA}\Omega_A^a \Omega_A^b - I_{ab,0}^{AB}\Omega_A^a \Omega_B^b + \frac{1}{2}I_{ab,0}^{BB}\Omega_B^a \Omega_B^b \right] \end{aligned} \quad (148)$$

and

$$\gamma_0 = -\Delta m \frac{N_c}{3} \sum_{i=1,2} \sum_n c_i \operatorname{sgn}(E_n) \operatorname{erfc}\left(\frac{|E_n|}{\Lambda_i}\right) \langle n | \gamma_0 | n \rangle, \quad (149)$$

$$I_{ab,0}^{AA} = \frac{N_c}{8} \sum_{i=1,2} \sum_{m,n} c_i f(E_m, E_n, \Lambda_i) \langle n | \lambda_a | m \rangle \langle m | \lambda_b | n \rangle, \quad (150)$$

$$I_{ab,0}^{AB} = \frac{N_c}{4} \sum_{i=1,2} \sum_{m,n} c_i f(E_m, E_n, \Lambda_i) \langle n | \lambda_a | m \rangle \langle m | J_b | n \rangle, \quad (151)$$

$$I_{ab,0}^{BB} = \frac{N_c}{2} \sum_{i=1,2} \sum_{m,n} c_i f(E_m, E_n, \Lambda_i) \langle n | J_a | m \rangle \langle m | J_b | n \rangle, \quad (152)$$

where  $c_1 = c, c_2 = 1 - c$  and the cutoff function  $f(E_m, E_n, \Lambda)$  is defined in Eq. (109). For the estimation of the imaginary part of the action, we need the following manipulation,

$$\begin{aligned} S_I &= -\frac{i}{2} N_c \operatorname{Sp} \log d(d^\dagger)^{-1} \\ &= \int dt \int \frac{d\omega}{2\pi} \operatorname{Tr} \log \left( \frac{w - H(U_S) - H_{SB} - \Omega_A + \Omega_B}{w - H(U_S) - H_{SB} + \Omega_A - \Omega_B} \right) \\ &= \int dt \int \frac{d\omega}{2\pi} \int_{-1}^1 d\lambda \operatorname{Tr} \left( \frac{-\Omega_A + \Omega_B}{w - H(U_S) - H_{SB} - \lambda(\Omega_A - \Omega_B)} \right). \end{aligned}$$

With replacement  $\omega - \lambda(\Omega_A - \Omega_B) \rightarrow \omega$  and analytical continuation for  $\omega$ , we obtain

$$\begin{aligned} &= \int dt \int \frac{d\omega}{2\pi} \int_{-1}^1 d\lambda \operatorname{Tr} \left( \frac{-\Omega_A + \Omega_B}{w - H(U_S) - H_{SB}} \right) \\ &= 2i \int \frac{d\omega'}{2\pi} \operatorname{Sp} \frac{(\Omega_A - \Omega_B)(H(U_S) + H_{SB})}{\omega'^2 + (H(U_S) + H_{SB})^2}, \quad \omega = i\omega'. \end{aligned} \quad (153)$$

Introducing the following proper-time regularization,

$$\frac{1}{\omega'^2 + (H(U_S) + H_{SB})^2} \rightarrow \int_{1/\Lambda^2}^\infty d\tau \exp[-\tau(\omega'^2 + (H(U_S) + H_{SB})^2)] \quad (154)$$

and expand up to first order, we finally obtain the form

$$S_I \rightarrow - \int dt \left[ \frac{\sqrt{3}}{2} B[U_S] \Omega_A^8 + K_{ab,0}^A D_{8a} \Omega_A^b + K_{ab,0}^B D_{8a} \Omega_B^b \right] \quad (155)$$

and

$$K_{ab,0}^A = \Delta m \frac{N_c}{4\sqrt{3}} \sum_{m,n} F(E_m, E_n, \Lambda) \langle n | \beta \lambda_a | m \rangle \langle m | \lambda_b | n \rangle, \quad (156)$$

$$K_{ab,0}^B = \Delta m \frac{N_c}{2\sqrt{3}} \sum_{m,n} F(E_m, E_n, \Lambda) \langle n | \beta \lambda_a | m \rangle \langle m | J_b | n \rangle. \quad (157)$$



The moments of inertia  $K_{ab}^A, K_{ab}^B$  are derived from imaginary part of the effective action thus need no regularization. The “cut-off” function  $F(E_m, E_n, \Lambda)$  becomes

$$F(E_m, E_n, \Lambda \rightarrow \infty) = \frac{\text{sgn}(E_m) - \text{sgn}(E_n)}{E_m - E_n}. \quad (158)$$

Also the valence quark contributions for the moments of inertia read

$$\gamma_{\text{val}} = \Delta m \frac{2N_c}{3} \langle \text{val} | \gamma_0 | \text{val} \rangle, \quad (159)$$

$$K_{ab, \text{val}}^A = \Delta m \frac{N_c}{\sqrt{3}} \sum_{n \neq \text{val}} \frac{\langle \text{val} | \beta \lambda_a | n \rangle \langle n | \lambda_b | \text{val} \rangle}{E_n - E_{\text{val}}}, \quad (160)$$

$$K_{ab, \text{val}}^B = \Delta m \frac{2N_c}{\sqrt{3}} \sum_{n \neq \text{val}} \frac{\langle \text{val} | \beta \lambda_a | n \rangle \langle n | J_b | \text{val} \rangle}{E_n - E_{\text{val}}}, \quad (161)$$

$$I_{ab, \text{val}}^{AA} = \frac{N_c}{2} \sum_{n \neq \text{val}} \frac{\langle \text{val} | \lambda_a | n \rangle \langle n | \lambda_b | \text{val} \rangle}{E_n - E_{\text{val}}}, \quad (162)$$

$$I_{ab, \text{val}}^{AB} = N_c \sum_{n \neq \text{val}} \frac{\langle \text{val} | \lambda_a | n \rangle \langle n | J_b | \text{val} \rangle}{E_n - E_{\text{val}}}, \quad (163)$$

$$I_{ab, \text{val}}^{BB} = 2N_c \sum_{n \neq \text{val}} \frac{\langle \text{val} | J_a | n \rangle \langle n | J_b | \text{val} \rangle}{E_n - E_{\text{val}}}. \quad (164)$$

Finally we can construct the following form for the effective lagrangian

$$\begin{aligned} L = & -E_{\text{static}} - \frac{\sqrt{3}}{2} B [U_S] \Omega_A^8 - \frac{1}{2} \gamma (1 - D_{88}) \\ & - K_{ab}^A D_{8a} \Omega_A^b - K_{ab}^B D_{8a} \Omega_B^b \\ & + \frac{1}{2} I_{ab}^{AA} \Omega_A^a \Omega_A^b - I_{ab}^{AB} \Omega_A^a \Omega_B^b + \frac{1}{2} I_{ab}^{BB} \Omega_B^a \Omega_B^b, \end{aligned} \quad (165)$$

and total moments of inertia acquire due to their sum, e.g.  $I_{ab}^{AA} = I_{ab, \text{val}}^{AA} + I_{ab, 0}^{AA}$ . where  $E_{\text{static}}$  is the self-consistent classical soliton energy and  $D_{ab}(A) = \frac{1}{2} \text{Tr}(\lambda_a A \lambda_b A^\dagger)$  is a  $SU(3)$  Wigner rotation matrix.

In the evaluations of the moments of inertia, the numerical difficulties may arise. Due to the difference of the boundary conditions between the initial and the final states of the matrix elements one may obtain the spurious nonzero values of the moments of inertia in the absence of background pion fields. Similar problem occurs in the  $\langle n | J_a | m \rangle$  for all values of  $a$  and  $\langle n | \lambda_a | m \rangle$  for  $a \geq 4$  with our harmonic oscillator basis. To avoid the problem, we employ the replacement of Eq. (112) for  $\langle n | J_a | m \rangle$  [15]. For the matrix element of  $\lambda_a$ , similar replacement induces an additional spurious term proportional to the mass difference. However, within our perturbative treatment in the mass difference, this procedure is justified.

With the standard canonical quantization formula for the collective coordinates we obtain the following quantization prescriptions

$$R_a = \begin{cases} -\sum_j (I_{aj}^{AA} \Omega_A^j - I_{aj}^{AB} \Omega_B^j - K_{aj}^A D_{8j}), & a = 1, 2, 3, \\ -\sum_b (I_{ab}^{AA} \Omega_A^b - K_{ab}^A D_{8b}), & a = 4, 5, 6, 7, \\ \frac{\sqrt{3}}{2} B, & a = 8, \end{cases} \quad (166)$$

and

$$K_i = -\sum_j (I_{ij}^{BB} \Omega_B^j - I_{ij}^{BA} \Omega_A^j + K_{ij}^B D_{8j}), \quad i = 1, 2, 3, \quad (167)$$

where  $R_a$  is the right isospin generator of  $SU(3)$ , and  $K_i$  represents the generator of the spatial rotation.

Due to the symmetry of the soliton, only the following elements of the moments of inertia survive:

$$\begin{aligned} I_{11}^{AA} &= I_{22}^{AA}, I_{11}^{BB} = I_{22}^{BB}, \\ I_{33}^{BB} &= m^2 I_{33}^{AA}, I_{33}^{AB} = I_{33}^{BA} = -m I_{33}^{AA}, \\ I_{44}^{AA} &= I_{55}^{AA} = I_{66}^{AA} = I_{77}^{AA}, \\ K_{11}^{AA} &= K_{22}^{AA}, \\ K_{33}^{BB} &= -m K_{33}^{AA}, \\ K_{44}^{AA} &= K_{55}^{AA} = K_{66}^{AA} = K_{77}^{AA}. \end{aligned} \quad (168)$$

Thus the hamiltonian becomes

$$\begin{aligned} H &= E_{\text{static}} + H_0 + H_1, \\ H_0 &= \frac{1}{2} \frac{1}{I_{44}^{AA}} \sum_{a=1}^7 R_a^2 + \frac{1}{2} \left( \frac{1}{I_{11}^{AA}} - \frac{1}{I_{44}^{AA}} \right) \sum_{i=1}^3 R_i^2 \\ &\quad + \frac{1}{2} \frac{1}{I_{11}^{BB}} \sum_{i=1}^3 K_i^2 + \frac{1}{2} \left( \frac{1}{I_{33}^{AA}} - \frac{1}{I_{11}^{AA}} - \frac{m^2}{I_{11}^{BB}} \right) R_3^2. \end{aligned} \quad (169)$$

In the evaluation of the  $H$ , we adopt a simple perturbative treatment with the mass difference  $\Delta m$  [59]. Up to first order of the  $\Delta m$ , the  $H_1$  is written as

$$\begin{aligned} H_1 &= \frac{1}{2} \gamma (1 - D_{88}) - \frac{K_{11}^A}{I_{11}^{AA}} (D_{81} R_1 + D_{82} R_2) \\ &\quad - \frac{K_{33}^A}{I_{33}^{AA}} D_{83} R_3 - \frac{K_{44}^A}{I_{44}^{AA}} \sum_{k=4}^7 D_{8k} R_k. \end{aligned} \quad (170)$$

The hamiltonian in Eq. (169) is diagonalized by using following collective wave functions of the nonperturbative part of the hamiltonian  $H_0$

$$\begin{aligned} \Psi_{YII_3, Y'NN_3, JJ_3}^{(n)}(A, B) &= \sqrt{\dim(n)} (-1)^{\frac{Y'}{2} + N_3} \\ &\times D_{YII_3, Y'NN_3}^{(n)*}(A) D_{J_3, -mN_3}^{J*}(B). \end{aligned} \quad (171)$$

With these bases, the matrix element reduces to the integral of the three Wigner matrices which can be easily evaluated from the  $SU(3)$  Clebsch-Gordan coefficients [59, 63]. The actual computations of the Clebsch-Gordan coefficients can be performed by using the numerical algorithm in Ref. [64].

Table 6: The various moments of inertia, with  $m_s = 149$  MeV.

	Valence	Vacuum	Total
$I_{11}^{AA} [\text{GeV}^{-1}]$	7.49	4.05	11.54
$I_{11}^{BB} [\text{GeV}^{-1}]$	11.19	5.46	16.65
$I_{33}^{AA} [\text{GeV}^{-1}]$	4.36	1.61	5.97
$I_{44}^{AA} [\text{GeV}^{-1}]$	1.64	1.26	2.90
$K_{11}^A$	0.285	$1.38 \times 10^{-4}$	0.285
$K_{33}^A$	0.297	$1.60 \times 10^{-3}$	0.298
$K_{44}^A$	0.255	$-1.05 \times 10^{-3}$	0.254
$\gamma [\text{MeV}]$	292.0	1098.8	1390.8

Our numerical calculations were performed with the constituent mass  $M = 400$  MeV. For a diagonalization problem of the Dirac hamiltonian, we used the deformed harmonic oscillator basis [65] which is described in detail in Appendix A. The self-consistent classical mass was obtained as  $E_{\text{static}} = 2406$  MeV which differs from the result of  $SU(2)$  sector. The difference arises due to the specific choice of the cutoff scheme in Eqs. (137)-(139). The values of various moments of inertia are listed in Table 6. The quantized states are coupled to the multiplets  $\{\overline{10}\}$ ,  $\{27\}$ ,  $\{35\}$ ,  $\{28\}$  corresponding to  $(p, q) = (0, 3), (2, 2), (4, 1), (6, 0)$  respectively. In Table 7, we show all the mass of the dibaryon states for the multiplets. The energy levels of the dibaryon states belonging to each multiplets are shown in Fig. 11.

In a pioneering work of the  $SU(3)$  collective quantization of the chiral soliton [66], it is pointed out that because of the constraint  $Y_R = 2$  which arises from the trivial embedding, some states in the constituent quark model are not allowed in the soliton solution. Hence the state we obtained is not the lowest state in the  $S = -2$  sector and the configuration of H-dibaryon may have not be an axially symmetric. On the other hand,  $(I, J) = (0, 0)$  channel in the  $S = -6$  sector, corresponding to di-Omega  $\Omega\Omega$  could have a rather deeper bound. According to the data of the  $B = 1$  hedgehog analysis (in Ref. [59]), we expect the binding energy about  $\sim 200$  MeV. This state is rather promising as a candidate of the axially symmetric dibaryon.

In the perturbative treatment of Eqs. (165),(169) we retained only linear terms for the mass difference  $\Delta m$  and used  $SU(3)$  symmetric wave functions in Eq. (171) for the ground state. For the  $B = 1$  hedgehog case, such leading order approximation induces discrepancy with experiment [59]. But it is possible to minimize it by incorporating appropriate higher-order effects in the  $\Delta m$  [67]. In our estimation of

Table 7: Absolute mass of the dibaryon (in MeV), with  $m_s = 149$  MeV.

Multiplet	$(S \ I \ J)$	Mass	Multiplet	$(S \ I \ J)$	Mass
$\overline{10}$	$(0 \ 0 \ 1)$	3255	35	$(0 \ 2 \ 1)$	3610
				$(-1 \ \frac{5}{2} \ 1)$	3965
	$(-1 \ \frac{1}{2} \ 1)$	3467		$(-1 \ \frac{3}{2} \ 1)$	3727
				$(-2 \ 2 \ 1)$	4034
	$(-2 \ 1 \ 1)$	3679		$(-2 \ 1 \ 1)$	3844
				$(-3 \ \frac{3}{2} \ 1)$	4103
	$(-3 \ \frac{3}{2} \ 1)$	3891		$(-3 \ \frac{1}{2} \ 1)$	3960
				$(-4 \ 1 \ 1)$	4172
				$(-4 \ 0 \ 1)$	4077
				$(-5 \ \frac{1}{2} \ 1)$	4241
27	$(0 \ 1 \ 0)$	3309	28	$(0 \ 3 \ 0)$	3969
	$(-1 \ \frac{3}{2} \ 0)$	3573		$(-1 \ \frac{5}{2} \ 0)$	4054
	$(-1 \ \frac{1}{2} \ 0)$	3453		$(-2 \ 2 \ 0)$	4139
	$(-2 \ 2 \ 0)$	3841		$(-2 \ 2 \ 0)$	4139
	$(-2 \ 1 \ 0)$	3678		$(-3 \ \frac{3}{2} \ 0)$	4224
	$(-2 \ 0 \ 0)$	3597		$(-4 \ 1 \ 0)$	4309
	$(-3 \ \frac{3}{2} \ 0)$	3904		$(-5 \ \frac{1}{2} \ 0)$	4393
	$(-3 \ \frac{1}{2} \ 0)$	3781		$(-6 \ 0 \ 0)$	4478
	$(-4 \ 1 \ 0)$	3966			

moments of inertia, in order to remove the spurious contributions, we employ ad hoc approximation for the matrix elements and it is justified only if we confine our calculations up to first order. We expect an extension of our scheme to the second order in  $\Delta m$  is also feasible.

## 6 Concluding Remarks

This article reports the theoretical framework and the numerical results of the chiral quark soliton model for higher baryon number solutions. The model was inspired by the instanton liquid model of the QCD vacuum and thus incorporates the basic features of QCD, e.g. the chiral symmetry and its breakdown accompanied by the appearance of the Goldstone bosons. This model provides correct observable as a nucleon including mass, electromagnetic value, spin carried by quarks, parton distributions and octet, decuplet  $SU(3)$  baryon spectra. For  $B > 1$ , different topological configurations from  $B = 1$  are needed to obtain the minimal energy solutions. For  $B = 2$ , we employed the axially symmetric configuration which produces the minimal energy configuration in the Skyrme model. We also investigated the

$B = 3, 4, 5$  solitons with axial symmetry. The solution exhibits doubly degenerate bound spectra of the one-quark orbits. This relatively large degeneracy confirms that the solutions are stable local minima. For  $B > 2$ , the Skyrme model predicts that the solutions have only discrete symmetries. According to the prediction, we studied the CQSM with the chiral fields of such platonic symmetries. The discrete crystal-like symmetries exhibit much complicated structure and the study of such configurations is rather formidable task. However the analysis becomes much simpler when we adopt the rational map ansatz to the chiral fields since with this ansatz the chiral fields are separable in polar coordinates and radial coordinate, which makes the numerical technique developed for  $B = 1$  applicable to find solutions with higher  $B$ . We showed that the baryon densities inherit the same discrete symmetries as the chiral fields and obtained various degenerate bound spectra of the valence quarks depending on the background of chiral field configurations. Evaluating the radial component of the baryon density, shell-like structure of the valence quark spectra was observed. The group theory should predict these level structures resulting from the symmetry of the background potential. In fact the degeneracy of the valence spectra are determined by the winding number of the chiral fields as well as the shape deformation (symmetry) of solitons. The four-fold degeneracy of the lowest states may be ascribed to the chiral symmetry  $SU(2)_L \times SU(2)_R$  of the hamiltonian. To get better understanding of the relation between the quark level structure and the winding number or the shape deformation, further analysis will be worth to be done in future.

Upon quantization, we computed zero-mode rotational corrections to the classical energy. The study of the Finkelstein-Rubinstein constraints indicates that the axially symmetric solution with even  $B$  has the same quantum number as the physically observed nuclei. Some of the states may be observed in experiments. For example, in the  $B = 2$ , we obtained  $I = 0, J = 1$  ( ${}^3S_1$  : deuteron) and  $I = 1, J = 0$  ( ${}^1S_0$ ) solutions. The energy of  ${}^3S_1$  is lower than the  ${}^1S_0$ . The order is in agreement with the experimental observations. For  $B = 4$ , the quantum number of the ground state  $I = 0, J = 0$  coincide with the observation. For odd  $B$ , the constraint of  $C_2$  in Eq. (126) seems to assure the validity of the ansatz. Indeed, it provides the ground state as  $I = J = 1/2$  for  $B = 3$  and as  $I = 1/2, J = 3/2$  for  $B = 5$  due to their shell-like structure, which coincide with physical observations. However, unfortunately the constraint in Eq. (124) forbid such states. Consequently, the axially symmetric solitons with odd  $B$  can appear only as excited states.

For an  $SU(3)$  extension of the model, we adopted the collective quantization scheme with a trivial embedding form for the chiral fields. In order to estimate the effects of the quark mass difference, we performed the naive perturbative method in terms of the mass difference. We obtained the dibaryonic spectrum coupled to the multiplets of  $\{\overline{10}\}$ ,  $\{27\}$ ,  $\{35\}$ ,  $\{28\}$ . The state we obtained is not the lowest state in the  $S = -2$  sector. The configuration of H-dibaryon may have not be an axially symmetric. On the other hand,  $(I, J) = (0, 0)$  channel in the  $S = -6$

sector, corresponding to di-Omega  $\Omega\Omega$  may have a rather deeper bound. This state is promising as a candidate of the axially symmetric dibaryon.

In our analysis, all obtained states seem to be deep bound states. Consideration of the Casimir effect is, however, necessary to determine which states are stable. For the  $B = 1$  skyrmion, the Casimir energies of the rotational and the translational zero modes were estimated by various authors. Their predictions for the total mass are around  $-0.5 \sim -1.3$  GeV [41, 42, 43, 44, 45]. For the  $B = 2$  torus, the Casimir energies have not been estimated yet. The thorough analysis of the Casimir effects is desired in order to examine the stability.

## A Numerical Basis

### A.1 Kahana-Ripka basis

The numerical method widely used in this model is based on the expansion of the Dirac spinor in terms of an appropriate orthogonal basis. The Kahana-Ripka basis [28] which was originally constructed for diagonalizing the hamiltonian with the chiral fields of  $B = 1$  hedgehog ansatz is a plane-wave finite basis. The basis is discretized by imposing an appropriate boundary condition on the radial wave functions at the radius  $r_{\max}$  chosen to be sufficiently larger than the soliton size. The basis is then made finite by including only those states with the momentum  $k$  as  $k < k_{\max}$ . The results should be, however, independent on  $r_{\max}$  and  $k_{\max}$ .

The hamiltonian with hedgehog ansatz commutes with the parity and the grand-spin operator given by

$$\mathbf{K} = \mathbf{j} + \boldsymbol{\tau}/2 = \mathbf{l} + \boldsymbol{\sigma}/2 + \boldsymbol{\tau}/2,$$

where  $\mathbf{j}, \mathbf{l}$  are respectively total angular momentum and orbital angular momentum. Accordingly, the angular basis can be written as

$$|(lj)KM\rangle = \sum_{j_3\tau_3} C_{jj_3\frac{1}{2}\tau_3}^{KM} \left( \sum_{m\sigma_3} C_{lm\frac{1}{2}\sigma_3}^{jj_3} |lm\rangle \left| \frac{1}{2}\sigma_3 \right\rangle \right) \left| \frac{1}{2}\tau_3 \right\rangle. \quad (172)$$

With this angular basis, the normalized eigenstates of the free hamiltonian in a

spherical box with radius  $r_{\max}$  can be constructed as follows:

$$\begin{aligned}
u_{KM}^{(1)} &= N_k \begin{pmatrix} ij_K(kr)|(KK + \frac{1}{2})KM\rangle \\ \Delta_k j_{K+1}(kr)|(K + 1K + \frac{1}{2})KM\rangle \end{pmatrix}, \\
u_{KM}^{(2)} &= N_k \begin{pmatrix} ij_K(kr)|(KK - \frac{1}{2})KM\rangle \\ -\Delta_k j_{K-1}(kr)|(K - 1K - \frac{1}{2})KM\rangle \end{pmatrix}, \\
u_{KM}^{(3)} &= N_k \begin{pmatrix} i\Delta_k j_K(kr)|(KK + \frac{1}{2})KM\rangle \\ -j_{K+1}(kr)|(K + 1K + \frac{1}{2})KM\rangle \end{pmatrix}, \\
u_{KM}^{(4)} &= N_k \begin{pmatrix} i\Delta_k j_K(kr)|(KK - \frac{1}{2})KM\rangle \\ j_{K-1}(kr)|(K - 1K - \frac{1}{2})KM\rangle \end{pmatrix}, \\
v_{KM}^{(1)} &= N_k \begin{pmatrix} ij_{K+1}(kr)|(K + 1K + \frac{1}{2})KM\rangle \\ -\Delta_k j_K(kr)|(KK + \frac{1}{2})KM\rangle \end{pmatrix}, \\
v_{KM}^{(2)} &= N_k \begin{pmatrix} ij_{K-1}(kr)|(K - 1K - \frac{1}{2})KM\rangle \\ \Delta_k j_K(kr)|(KK - \frac{1}{2})KM\rangle \end{pmatrix}, \\
v_{KM}^{(3)} &= N_k \begin{pmatrix} i\Delta_k j_{K+1}(kr)|(K + 1K + \frac{1}{2})KM\rangle \\ j_K(kr)|(KK + \frac{1}{2})KM\rangle \end{pmatrix}, \\
v_{KM}^{(4)} &= N_k \begin{pmatrix} i\Delta_k j_{K-1}(kr)|(K - 1K - \frac{1}{2})KM\rangle \\ -j_K(kr)|(KK - \frac{1}{2})KM\rangle \end{pmatrix}, \tag{173}
\end{aligned}$$

with

$$N_k = \left[ \frac{1}{2} r_{\max}^3 \left( j_{K+1}(kr_{\max}) \right)^2 \right]^{-1/2} \tag{174}$$

and  $\Delta_k = k/(E_k + M)$ .

The momenta are discretized by the boundary condition  $j_K(k_i r_{\max}) = 0$ . The  $u, v$  correspond to the “natural” and “unnatural” components of the basis which stand for parity  $(-1)^K$  and  $(-1)^{K+1}$  respectively.

Let us construct the trial function using the Kahana-Ripka basis to solve the eigenequations in Eq. (7),

$$\begin{aligned}
\phi_\mu(\mathbf{x}) &= \lim_{K_{\max} \rightarrow \infty} \sum_{K=0}^{K_{\max}} \sum_{M=-K}^K \sum_{j=1}^4 [\alpha_{KM,\mu}^{(j)} u_{KM}^{(j)}(r, \theta, \varphi) \\
&\quad + \beta_{KM,\mu}^{(j)} v_{KM}^{(j)}(r, \theta, \varphi)]. \tag{175}
\end{aligned}$$

## A.2 Deformed Oscillator basis

Let us show the numerical analysis of the eigen equations in detail. To solve the eigenequation of the form,

$$[-i\boldsymbol{\alpha} \cdot \nabla + \beta M(\cos F(\rho, z) + i\gamma_5 \boldsymbol{\tau} \cdot \hat{\mathbf{n}}_R \sin F(\rho, z))]\phi_\mu(\mathbf{x}) = E_\mu \phi_\mu(\mathbf{x}), \tag{176}$$

we introduce the deformed harmonic oscillator spinor basis which was originally constructed by Gambhir *et al.* in the relativistic mean field theory for deformed nuclei [65]. The upper and lower components of the Dirac spinors are expanded separately by the basis as

$$\phi_\mu(\mathbf{x}) = \begin{pmatrix} f_\mu(\mathbf{x}) \\ ig_\mu(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \sum_a f_{\mu a} \Phi_a(\mathbf{x}, s) \\ i \sum_{\tilde{a}} g_{\mu \tilde{a}} \Phi_{\tilde{a}}(\mathbf{x}, s) \end{pmatrix} \chi_{m_\tau}^I \quad (177)$$

where  $\Phi_a(\mathbf{x}, s, \tau)$  are the eigefunctions of a deformed harmonic oscillator potential

$$V_{osc}(\rho, z) = \frac{1}{2} \mathcal{M} \omega_\rho^2 \rho^2 + \frac{1}{2} \mathcal{M} \omega_z^2 z^2, \quad (178)$$

and defined by

$$\Phi_a(\mathbf{x}, m_s) = \frac{1}{\sqrt{2\pi}} \phi_{n_r}^{|\omega|}(\rho) \phi_{n_z}(z) e^{i\omega\varphi} \chi_{m_s}^S \quad (179)$$

with

$$\begin{aligned} \phi_{n_r}^{|\omega|}(\rho) &= N_{n_r}^{|\omega|} (\sqrt{\alpha_\rho \rho})^{|\omega|} e^{-\frac{1}{2} \alpha_\rho \rho^2} L_{n_r}^{|\omega|}(\alpha_\rho \rho^2) \\ n_r &= 0, 1, 2, \dots, N_{\text{rmax}} \\ \phi_{n_z}(z) &= N_{n_z} e^{-\frac{1}{2} \alpha_z z^2} H_{n_z}(\sqrt{\alpha_z} z) \\ n_z &= 1, 3, \dots, 2N_{\text{zmax}} + 1 \quad \text{or} \quad 0, 2, \dots, 2N_{\text{zmax}}, \end{aligned}$$

and

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (180)$$

depending on if the eigenvalues of the third components of the spin  $m_s$  (isospin  $m_\tau$ ) takes  $+1$  or  $-1$ . The functions,  $L_{n_r}^{|\omega|}$  and  $H_{n_z}$ , are the associated Laguerre polynomials and the Hermite polynomials with the normalization constants

$$N_{n_r}^{|\omega|} = \sqrt{\frac{2\alpha_\rho n_r!}{(n_r + |\omega|)!}}, \quad N_{n_z} = \frac{1}{\sqrt{2^{n_z} n_z! \sqrt{\frac{\pi}{\alpha_z}}}}. \quad (181)$$

These polynomials can be calculated by following recursion relations

$$x \frac{d}{dx} L_n^m(x) = n L_n^m(x) - (n + m) L_{n-1}^m(x) \quad (182)$$

$$L_n^{m-1}(x) = L_n^m(x) - L_{n-1}^m(x) \quad (183)$$

and

$$H_{n+1}(x) - 2x H_n(x) + 2n_z H_{n-1}(x) = 0 \quad (184)$$

$$\frac{d}{dx} H_n(x) = 2n H_{n-1}(x) \quad (185)$$



where constants  $\alpha_\rho$  and  $\alpha_z$  can be expressed by the oscillator frequencies as

$$\alpha_\rho = \frac{\mathcal{M}\omega_\rho}{\hbar}, \quad \alpha_z = \frac{\mathcal{M}\omega_z}{\hbar} \quad (186)$$

which are free parameters chosen optimally. The  $N_{\text{rmax}}$  and  $N_{\text{zmax}}$  are increased until convergence is attained. The parity transformation rule of  $\Phi_\alpha$  is given by

$$\Phi_\alpha(\rho, \varphi + \pi, -z; s, t) = (-1)^{\omega+n_z} \Phi_\alpha(\rho, \varphi, z; s, t) \quad (187)$$

where

$$H_{n_z}(-\sqrt{\alpha_z}z) = (-1)^{n_z} H_{n_z}(\sqrt{\alpha_z}z), \quad (188)$$

has been used. The parity is  $+$  for  $\omega + n_z = \text{odd}$ , and  $-$  for  $\omega + n_z = \text{even}$ .

There are two sets of the complete basis for each parity. One is the natural basis with  $K_3^P = 0^+, 1^-, 2^+, \dots$ , for odd  $B$  and  $K_3^P = \frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+, \dots$  for even  $B$ . Another is the unnatural basis with  $K_3^P = 0^-, 1^+, 2^-, \dots$ , for odd  $B$  and  $K_3^P = \frac{1}{2}^-, \frac{3}{2}^+, \frac{5}{2}^-, \dots$  for even  $B$ . The natural basis is given by

$$\begin{aligned} \phi_\mu^n(\mathbf{x}) = & \left( \begin{array}{c} \sum_{\alpha(0)} f_{\alpha(0),\mu} \Phi_{\alpha(0)}(\mathbf{x}, \uparrow_S) + \sum_{\alpha(1)} f_{\alpha(1),\mu} \Phi_{\alpha(1)}(\mathbf{x}, \downarrow_S) \\ i \sum_{\beta(0)} g_{\beta(0),\mu} \Phi_{\beta(0)}(\mathbf{x}, \uparrow_S) + i \sum_{\beta(1)} g_{\beta(1),\mu} \Phi_{\beta(1)}(\mathbf{x}, \downarrow_S) \end{array} \right) \chi_u^I \\ & + \left( \begin{array}{c} \sum_{\alpha(2)} f_{\alpha(2),\mu} \Phi_{\alpha(2)}(\mathbf{x}, \uparrow_S) + \sum_{\alpha(3)} f_{\alpha(3),\mu} \Phi_{\alpha(3)}(\mathbf{x}, \downarrow_S) \\ i \sum_{\beta(2)} g_{\beta(2),\mu} \Phi_{\beta(2)}(\mathbf{x}, \uparrow_S) + i \sum_{\beta(3)} g_{\beta(3),\mu} \Phi_{\beta(3)}(\mathbf{x}, \downarrow_S) \end{array} \right) \chi_d^I \end{aligned} \quad (189)$$

where

$$\begin{aligned} \alpha(0) &= \{n_r, n_z : \text{odd}, \omega_0 \equiv K_3 - 1/2 - m_w/2\} \\ \alpha(1) &= \{n_r, n_z : \text{even}, \omega_1 \equiv K_3 + 1/2 - m_w/2\} \\ \alpha(2) &= \{n_r, n_z : \text{even}, \omega_2 \equiv K_3 - 1/2 + m_w/2\} \\ \alpha(3) &= \{n_r, n_z : \text{odd}, \omega_3 \equiv K_3 + 1/2 + m_w/2\} \end{aligned}$$

and

$$\begin{aligned} \beta(0) &= \{n_r, n_z : \text{even}, \omega_0 \equiv K_3 - 1/2 - m_w/2\} \\ \beta(1) &= \{n_r, n_z : \text{odd}, \omega_1 \equiv K_3 + 1/2 - m_w/2\} \\ \beta(2) &= \{n_r, n_z : \text{odd}, \omega_2 \equiv K_3 - 1/2 + m_w/2\} \\ \beta(3) &= \{n_r, n_z : \text{even}, \omega_3 \equiv K_3 + 1/2 + m_w/2\}. \end{aligned}$$

The unnatural basis  $\phi_\mu^{(u)}$  is given by replacing,  $\alpha \leftrightarrow \beta$  in Eq. (189). By using the natural and unnatural basis, the eigenvalue problem in Eq. (176) can be reduced to a symmetric matrix diagonalization problem.

Let us calculate the matrix elements of the hamiltonian below. For the kinetic term  $\boldsymbol{\alpha} \cdot \mathbf{p}$ , we have

$$\begin{aligned}
& \langle \Phi_{\alpha(0)} | \boldsymbol{\sigma} \cdot \mathbf{p} | i \Phi_{\beta'(0)} \rangle \\
&= \frac{1}{2\pi} \int d^3x \phi_{n_r}^{|\omega_0|}(\rho) \phi_{n_z}(z) e^{-i\omega_0\varphi} \left( \frac{\partial}{\partial z} \right) \phi_{n'_r}^{|\omega'_0|}(\rho) \phi_{n'_z}(z) e^{i\omega'_0\varphi} \\
&= \delta_{n_r n'_r} (N_{n_z} N_{n'_z} \sqrt{\alpha_z} n'_z \frac{1}{N_{n_z}^2} \delta_{n_z n'_z-1} - \frac{1}{2} N_{n_z} N_{n'_z} \sqrt{\alpha_z} \frac{1}{N_{n_z}^2} \delta_{n_z n'_z+1}) \\
&= \begin{cases} \delta_{\omega_0 \omega'_0} \delta_{n_r n'_r} \frac{N'_{n_z}}{N_{n_z}} \sqrt{\alpha_z} n'_z \delta_{n_z n'_z-1} \\ \delta_{\omega_0 \omega'_0} \delta_{n_r n'_r} \left(-\frac{1}{2}\right) \frac{N'_{n_z}}{N_{n_z}} \sqrt{\alpha_z} \delta_{n_z n'_z+1}, \end{cases} \quad (190)
\end{aligned}$$

$$\begin{aligned}
& \langle \Phi_{\alpha(0)} | \boldsymbol{\sigma} \cdot \mathbf{p} | i \Phi_{\beta'(1)} \rangle \\
&= \frac{1}{2\pi} \int d^3x \phi_{n_r}^{|\omega_0|}(\rho) \phi_{n_z}(z) e^{-i\omega_0\varphi} e^{-i\varphi} \left( \frac{\partial}{\partial \rho} - \frac{i}{\rho} \frac{\partial}{\partial \varphi} \right) \phi_{n'_r}^{|\omega'_1|}(\rho) \phi_{n'_z}(z) e^{i\omega'_1\varphi} \\
&= \begin{cases} \delta_{n_z n'_z} \sqrt{\alpha_r} (\sqrt{n_r + \omega_0 + 1} \delta_{n_r n'_r} + \sqrt{n_r} \delta_{n_r-1 n'_r}) \\ (\omega_0 \geq 0 : \omega'_1 = \omega_0 + 1 > 0) \\ -\delta_{n_z n'_z} \sqrt{\alpha_r} (\sqrt{n_r - \omega_0} \delta_{n_r n'_r} + \sqrt{n_r + 1} \delta_{n_r n'_r-1}). \\ (\omega_0 < 0 : \omega'_1 = \omega_0 + 1 \leq 0) \end{cases} \quad (191)
\end{aligned}$$

In the natural basis, quantum numbers  $(n_z, n'_z)$  takes values  $(1, 2), (3, 4), \dots$  for the upper part and  $(1, 0), (3, 2), \dots$  for the lower part. In the unnatural basis,  $(n_z, n'_z) = (0, 1), (2, 3), \dots$  for the upper part and  $(n_z, n'_z) = (2, 1), (3, 2), \dots$  for the lower part.

For the potential term  $\beta M(\cos F(\rho, z) + i\gamma_5 \boldsymbol{\tau} \cdot \hat{\mathbf{n}}_R \sin F(\rho, z))$ , we have

$$\begin{aligned}
& \langle \Phi_{\alpha(0)} \chi_u^I | M \cos F(\rho, z) | \Phi_{\alpha'(0)} \chi_u^I \rangle \\
&= \int \rho d\rho dz M \cos F(\rho, z) \phi_{n_r}^{|\omega_0|}(\rho) \phi_{n_z}(z) \phi_{n'_r}^{|\omega_0|}(\rho) \phi_{n'_z}(z), \quad (192)
\end{aligned}$$

$$\begin{aligned}
& \langle \Phi_{\alpha(0)} \chi_u^I | M i \boldsymbol{\tau} \cdot \hat{\mathbf{n}}_R \sin F(\rho, z) | i \Phi_{\beta'(0)} \chi_u^I \rangle \\
&= - \int \rho d\rho dz M \cos \Theta(\rho, z) \sin F(\rho, z) \phi_{n_r}^{|\omega_0|}(\rho) \phi_{n_z}(z) \phi_{n'_r}^{|\omega'_0|}(\rho) \phi_{n'_z}(z), \quad (193)
\end{aligned}$$

$$\begin{aligned}
& \langle \Phi_{\alpha(0)} \chi_u^I | M i \boldsymbol{\tau} \cdot \hat{\mathbf{n}}_R \sin F(\rho, z) | \Phi_{\beta'(2)} \chi_d^I \rangle \\
&= - \int \rho d\rho dz M \sin \Theta(\rho, z) \sin F(\rho, z) \phi_{n_r}^{|\omega_0|}(\rho) \phi_{n_z}(z) \phi_{n'_r}^{|\omega_2|}(\rho) \phi_{n'_z}(z). \quad (194)
\end{aligned}$$

Other elements can be calculated in the same manner.

To estimate the moments of inertia, *e.g.*,  $\langle n | \lambda_a | m \rangle$  let us introduce the basis for the strange direction [61]. The hamiltonian for the strange quark commutes with

the total angular momentum

$$J_3 = L_3 + \frac{1}{2}\sigma_3 \quad (195)$$

because of the trivial construction of the  $SU(3)$  chiral fields in Eq. (141). As a result, the deformed basis has common form with the  $SU(2)$  (see Eqs. (179)-(189)), except in the quantum number in Eq. (189) as

$$\begin{aligned} \alpha(0) &= \{n_r, n_z : \text{even}, \omega_0 \equiv J_3 - 1/2\} \\ \alpha(1) &= \{n_r, n_z : \text{odd}, \omega_1 \equiv J_3 + 1/2\} \end{aligned}$$

and

$$\begin{aligned} \beta(0) &= \{n_r, n_z : \text{odd}, \omega_0 \equiv J_3 - 1/2\} \\ \beta(1) &= \{n_r, n_z : \text{even}, \omega_1 \equiv J_3 + 1/2\}. \end{aligned}$$

The unnatural basis  $\phi_\mu^{(u)}$  is given by replacing,  $\alpha \leftrightarrow \beta$  in Eq. (189).

## B Operation $\hat{R}$ to the Numerical Basis

In this appendix, we present the construction of an operator  $\hat{R}$  to the Kahana-Ripka basis in detail. Since it operates on the spherical harmonics, let us write the spherical harmonics in terms of the complex variables  $z$  and their conjugate  $\bar{z}$  as follows :

$$z = \tan \frac{\theta}{2} e^{i\varphi}, \quad (196)$$

which can be related to the usual polar coordinates by

$$\cos \theta = \frac{1 - |z|^2}{1 + |z|^2}, \quad \exp(i\varphi) = \pm \sqrt{\frac{z}{\bar{z}}} \quad (197)$$

According to the definition of the spherical harmonics [68], we obtain (up to  $l \leq 3$ )

$$Y_{11} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{2z}{1+|z|^2}, \quad Y_{10} = \frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{1-|z|^2}{1+|z|^2}, \quad Y_{1-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{2\bar{z}}{1+|z|^2}, \quad (198)$$

$$Y_{22} = \frac{1}{4}\sqrt{\frac{3 \cdot 5}{2\pi}}\left(\frac{2z}{1+|z|^2}\right)^2, \quad Y_{21} = -\frac{1}{2}\sqrt{\frac{3 \cdot 5}{2\pi}}\frac{2z(1-|z|^2)}{(1+|z|^2)^2},$$

$$Y_{20} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3\left(\frac{1-|z|^2}{1+|z|^2}\right)^2 - 1\right), \quad (199)$$

$$Y_{2-1} = \frac{1}{2}\sqrt{\frac{3 \cdot 5}{2\pi}}\frac{2\bar{z}(1-|z|^2)}{(1+|z|^2)^2}, \quad Y_{2-2} = \frac{1}{4}\sqrt{\frac{3 \cdot 5}{2\pi}}\left(\frac{2\bar{z}}{1+|z|^2}\right)^2,$$

$$Y_{33} = -\frac{1}{8}\sqrt{\frac{5 \cdot 7}{\pi}}\left(\frac{2z}{1+|z|^2}\right)^3, \quad Y_{32} = \frac{1}{4}\sqrt{\frac{3 \cdot 5 \cdot 7}{2\pi}}\frac{4z^2(1-|z|^2)}{(1+|z|^2)^3},$$

$$Y_{31} = -\frac{1}{8}\sqrt{\frac{3 \cdot 7}{\pi}}\left(5\left(\frac{1-|z|^2}{1+|z|^2}\right)^2 - 1\right)\frac{2z}{1+|z|^2},$$

$$Y_{30} = \frac{1}{4}\sqrt{\frac{7}{\pi}}\left(5\left(\frac{1-|z|^2}{1+|z|^2}\right)^2 - 3\right)\frac{1-|z|^2}{1+|z|^2}, \quad (200)$$

$$Y_{3-1} = \frac{1}{8}\sqrt{\frac{3 \cdot 7}{\pi}}\left(5\left(\frac{1-|z|^2}{1+|z|^2}\right)^2 - 1\right)\frac{2\bar{z}}{1+|z|^2},$$

$$Y_{3-2} = \frac{1}{4}\sqrt{\frac{3 \cdot 5 \cdot 7}{2\pi}}\frac{4\bar{z}^2(1-|z|^2)}{(1+|z|^2)^3}, \quad Y_{3-3} = \frac{1}{8}\sqrt{\frac{5 \cdot 7}{\pi}}\left(\frac{2\bar{z}}{1+|z|^2}\right)^3.$$

According to the (inverse) transformation of the  $Z_2 \times Z_2$

$$z \rightarrow -\frac{1}{z} \quad \equiv \quad \theta \rightarrow \pi - \theta, \varphi \rightarrow \pi - \varphi, \quad (201)$$

the spherical harmonics are transformed via

$$Y_{lm}(\theta, \varphi) \rightarrow Y_{lm}(\pi - \theta, \pi - \varphi) = (-1)^{l-m} Y_{l-m}(\theta, \varphi). \quad (202)$$

For the (inverse) tetrahedral transformation, we have

$$z \rightarrow i \frac{1-z}{1+z}, \quad (203)$$

Thus the transformation of the spherical harmonics is given by

$$\begin{aligned}
Y_{11} &\rightarrow -\frac{i}{2}Y_{11} - \frac{i}{\sqrt{2}}Y_{10} - \frac{i}{2}Y_{1-1}, \\
Y_{10} &\rightarrow -\frac{i}{\sqrt{2}}Y_{11} + \frac{i}{\sqrt{2}}Y_{1-1}, \\
Y_{1-1} &\rightarrow \frac{i}{2}Y_{11} + \frac{i}{\sqrt{2}}Y_{10} + \frac{i}{2}Y_{1-1}.
\end{aligned} \tag{204}$$

$$\begin{aligned}
Y_{22} &\rightarrow -\frac{1}{4}Y_{22} - \frac{1}{2}Y_{21} - \frac{1}{2}\sqrt{\frac{3}{2}}Y_{20} - \frac{1}{2}Y_{2-1} - \frac{1}{4}Y_{2-2}, \\
Y_{21} &\rightarrow \frac{i}{\sqrt{2}}Y_{22} + \frac{i}{\sqrt{2}}Y_{21} - \frac{i}{2}Y_{2-1} - \frac{i}{2}Y_{2-2}, \\
Y_{20} &\rightarrow \frac{1}{2}\sqrt{\frac{3}{2}}Y_{22} - \frac{1}{2}Y_{20} + \frac{1}{2}\sqrt{\frac{3}{2}}Y_{2-2}, \\
Y_{2-1} &\rightarrow -\frac{i}{\sqrt{2}}Y_{22} + \frac{i}{\sqrt{2}}Y_{21} - \frac{i}{2}Y_{2-1} + \frac{i}{2}Y_{2-2}, \\
Y_{2-2} &\rightarrow -\frac{1}{4}Y_{22} + \frac{1}{2}Y_{21} - \frac{1}{2}\sqrt{\frac{3}{2}}Y_{20} + \frac{1}{2}Y_{2-1} - \frac{1}{4}Y_{2-2}.
\end{aligned} \tag{205}$$

$$\begin{aligned}
Y_{33} &\rightarrow \frac{i}{8}Y_{33} + \frac{\sqrt{6}i}{8}Y_{32} + \frac{\sqrt{15}i}{8}Y_{31} + \frac{\sqrt{5}i}{4}Y_{30} + \frac{\sqrt{15}i}{8}Y_{3-1} + \frac{\sqrt{6}i}{8}Y_{3-2} + \frac{i}{8}Y_{3-3}, \\
Y_{32} &\rightarrow \frac{1}{4}\sqrt{\frac{3}{2}}Y_{33} + \frac{1}{2}Y_{32} + \frac{1}{4}\sqrt{\frac{5}{2}}Y_{31} - \frac{1}{4}\sqrt{\frac{5}{2}}Y_{3-1} - \frac{1}{2}Y_{3-2} - \frac{1}{4}\sqrt{\frac{3}{2}}Y_{3-3}, \\
Y_{31} &\rightarrow -\frac{\sqrt{15}i}{8}Y_{33} - \frac{\sqrt{10}i}{8}Y_{32} + \frac{i}{8}Y_{31} + \frac{\sqrt{3}i}{4}Y_{30} + \frac{i}{8}Y_{3-1} - \frac{\sqrt{10}i}{8}Y_{3-2} - \frac{\sqrt{15}i}{8}Y_{3-3}, \\
Y_{30} &\rightarrow -\frac{\sqrt{5}}{4}Y_{33} + \frac{\sqrt{3}}{4}Y_{31} - \frac{\sqrt{3}}{4}Y_{3-1} + \frac{\sqrt{5}}{4}Y_{3-3}, \\
Y_{3-1} &\rightarrow \frac{\sqrt{15}i}{8}Y_{33} - \frac{\sqrt{10}i}{8}Y_{32} - \frac{i}{8}Y_{31} + \frac{\sqrt{3}i}{4}Y_{30} - \frac{i}{8}Y_{3-1} - \frac{\sqrt{10}i}{8}Y_{3-2} + \frac{\sqrt{15}i}{8}Y_{3-3}, \\
Y_{3-2} &\rightarrow \frac{1}{4}\sqrt{\frac{3}{2}}Y_{33} - \frac{1}{2}Y_{32} + \frac{1}{4}\sqrt{\frac{5}{2}}Y_{31} - \frac{1}{4}\sqrt{\frac{5}{2}}Y_{3-1} + \frac{1}{2}Y_{3-2} - \frac{1}{4}\sqrt{\frac{3}{2}}Y_{3-3}, \\
Y_{3-3} &\rightarrow -\frac{i}{8}Y_{33} + \frac{\sqrt{6}i}{8}Y_{32} - \frac{\sqrt{15}i}{8}Y_{31} + \frac{\sqrt{5}i}{4}Y_{30} - \frac{\sqrt{15}i}{8}Y_{3-1} + \frac{\sqrt{6}i}{8}Y_{3-2} - \frac{i}{8}Y_{3-3}.
\end{aligned} \tag{206}$$

The operation  $\hat{R}$  to the basis becomes

$$\hat{R}|(lj)KM\rangle \equiv \hat{K} \sum_{j_3\tau_3} C_{jj_3\frac{1}{2}\tau_3}^{KM} \left( \sum_{m\sigma_3} C_{lm\frac{1}{2}\sigma_3}^{jj_3} |lm\rangle' \left| \frac{1}{2}\sigma_3 \right\rangle \right) \left| \frac{1}{2}\tau_3 \right\rangle, \tag{207}$$

where  $|lm\rangle'$  represents the transformations of the spherical harmonics in Eqs. (202) or (205)-(207). Evaluating  $\hat{R}$  for each symmetric operation, one can get the final results in Eqs. (93)-(97).

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Figure 1: Spectra of the quark orbits of  $B = 2$  and  $B = 4$  with axially symmetric ansatz (23), as a function of the soliton size parameter  $X$  [21].

Figure 2: Contour plot of the profile functions  $F(\rho, z), \Theta(\rho, z)$ , of  $B = 2, 3$  with axial symmetry [21].

Figure 3: Contour plot of the profile functions  $F(\rho, z), \Theta(\rho, z)$ , of  $B = 4, 5$  with axial symmetry [21].

Figure 4: Contour plot of the baryon number densities  $b(\boldsymbol{x})$  [ $\text{fm}^{-3}$ ] (37) with axial symmetry [21].

Figure 5: Self-consistent profile functions for  $B = 3 - 9, 17$  in the rational map ansatz calculations [20].

Figure 6: Surface plot of the baryon number densities  $b(\boldsymbol{x})$  (45) for  $B = 3 - 9$  and the excited states  $B = 5^*, 9^*$  [20].

Figure 7: Valence quark spectra for  $B = 1 - 9, 17$  [20].

Figure 8: Angular averaged baryon densities of  $i$ th valence quarks  $\rho^{(i)}(r)$  of  $B = 3 - 9, 17$ , with the occupation number and the eigenvalue (in MeV) [20].

Figure 9: Surface plot of the baryon number densities for  $B = 2$  configurations (50) [49].

Figure 10: Total energies of the  $B = 2$  configurations [49].

Figure 11: The dibaryon spectra [15].

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